

# Nonequilibrium Multi-Band Spin Quantum Transport Equations: Spin, Pseudo-Spin, and Total Charge Coupling

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## Abstract

Using the superfield nonequilibrium Green's function technique, we derive the spatio-temporal spin magnetization quantum transport equations (SMQTEs) for a two-band model of semiconductors. The relevant variables are the real (Pauli-Dirac) spin, pseudo-spin, and the total charge. The results show that the multi-band real SMQTEs are coupled to the pseudo-spin magnetization transport equations by virtue of the presence of two additional discrete quantum labels besides the up and down real-spin indices, namely, the conduction and valence band quantum labels. The SMQTEs essentially consist of three group of terms describing the rate of change, namely, (1) a group of terms similar to the equation for particle quantum transport, i.e., with spin-independent transport parameters, (2) a group of terms describing various torques influencing the spin orientation and directional flow of spin magnetization correlations or phase-space magnetization density, and (3) a group of terms expressing the coupling of the real spin magnetization with the pseudo-spin magnetization. Self-consistently, the pseudo-spin magnetization equations incorporate the pseudo-spin/real spin coupling, as well as the pseudo-spin coupling to the total charge.

## I. INTRODUCTION

The need to analyze the ultrafast-switching-speed and power-dissipation (speed-power product) performance of nanoelectronic devices has ushered the extension of the classical Boltzmann transport equation to a fully time-dependent and highly-nonlinear nonequilibrium quantum distribution function (QDF) transport equations for charge carriers. This has been achieved through the use of non-equilibrium Green's function, obtained either by the time-contour quantum field formulation of Schwinger<sup>1</sup>, Keldysh<sup>2</sup>, and Kadanoff and Baym<sup>3</sup>, or by the real-time quantum superfield formulation of Buot, coupled with his lattice Weyl (LW) transformation technique.<sup>4-7</sup> This extension to QDF transport equation has proved to be highly crucial in discovering autonomous THz current oscillations in resonant tunneling devices through numerical simulations, and in resolving controversial issues concerning the highly-nonlinear and bistable current-voltage characteristics found in the experiments<sup>8,9</sup>. Indeed, in the phase space QDF kinetic approach has so far been the most successful technique in the time-dependent analyses of open and active nanosystem and nanodevices, as evidenced by the numerical work of Jensen and Buot on resonant tunneling heterostructures<sup>9</sup>.

To the authors' knowledge, the extension of the classical spin density equation or the Bloch equation for spin transport<sup>10</sup>, which is the analogue of the classical Boltzmann equation for charged-particle transport, to fully space-time dependent and highly-nonlinear nonequilibrium QDF transport equations for the magnetization, has only been recently reported by the author for a single energy band in a co-authored paper<sup>11</sup>. With the exploding surge of interest on spintronics and nanomagnetism<sup>12</sup>, there is an urgent need for this fully quantum transport extension of the classical Bloch equation to guide the time-dependent numerical simulation of the speed/power switching performance and reliability analyses of realistic functional spin nanostructures and transistors. In this paper, we report the multiband SMQTEs.

An immense activity with time-dependent spin magnetization equation, driven by technological applications, occurs in the field of micromagnetics dealing mostly with micro-domain wall dynamics. The so-called classical Landau-Lifshitz-Gilbert equation, with its various modifications and phenomenological damping terms, has become the major player in the analyses of the dynamics of the magnetization field.<sup>13-15</sup>

In magnetoelectronics and spintronics, the mutual dependence of electron transport prop-

erties and magnetic properties have resulted in various phenomena referred to as giant magnetoresistance<sup>16</sup>, spin modulator<sup>17</sup> based on spin precession with or without magnetic field (based on spin-orbit coupling), and spin transfer torque<sup>18,19</sup> which have become springboards of several novel spin-based device concepts.

In analyzing the current-induced spin orientation of electrons in semiconductors, with spin-orbit coupling, Dyakonov and Perel<sup>10</sup> used the following transport equation for the spin spatial-density vector,  $\vec{S}$ , as

$$\frac{\partial}{\partial t}\vec{S} = \left[ -\vec{\nabla} \cdot \vec{Q} - \frac{\vec{S}}{\tau_s} \right] + \vec{\Omega} \times \vec{S}, \quad (1)$$

where  $\vec{Q}$  is the spin flux density,

$$\vec{\Omega} = \frac{1}{\hbar} \mu_B g \vec{B},$$

$\mu_B$  is the Bohr magneton,  $g$  is the g-factor for electrons, and  $\vec{B}$  is the effective magnetic field. Earlier, in the 1950's Torrey<sup>20</sup> employed the classical Bloch equations for spin systems in the diffusive regime as,

$$\frac{\partial \vec{M}}{\partial t} = \left[ D \nabla^2 \vec{M} - \vec{M}_\tau \right] + \gamma \vec{M} \times \vec{B}, \text{ where } \vec{M}_\tau = \left\{ \frac{M_x}{\tau_2}, \frac{M_y}{\tau_2}, \frac{M_z}{\tau_1} \right\}, \quad (2)$$

where  $D$  is the diffusion coefficient,  $\tau_1$  is the spin-magnetization relaxation characteristic time, and  $\tau_2$  is the spin-dephasing characteristic time for the  $M_x$  and  $M_y$  components. Most of the works that follows on spin transport also make use of classical spin transport equations. These classical techniques are no longer valid for analyses of the ultrafast switching speed and power-dissipation performance of the emerging spintronic devices.

What these classical treatments have earlier shown is that spin transport consist of terms similar to particle transport, i.e., with spin-independent transport parameters, such as the diffusion coefficient,  $D$ , and relaxation times in Eq. (2), and terms describing the torques in the system.<sup>21</sup> However, the spin-charge coupling for a single band transport<sup>22</sup>, and spin/pseudo-spin/charge interaction<sup>23</sup> in multiband transport are lacking in the equations or at best not fully treated self-consistently within these classical and semiclassical treatments.<sup>24</sup>

The purpose of this paper is to extend the above classical magnetization transport equations to a fully time-dependent and highly-nonlinear nonequilibrium spin magnetization QDF transport equations in phase space, based on nonequilibrium Green's function

technique,<sup>1-5,7</sup> on the same level as in the extension of the classical Boltzmann kinetic equation to full QDF transport equation for electrons. In the single-band magnetization QDF transport equations, the spin-charge coupling is explicitly incorporated. In our present multi-band generalization, we found that besides the highly-coupled transport equations for the Pauli-Dirac or real-spin density, nonlinear coupling to the pseudo-spin density is also present in all of the spin magnetization quantum transport equations for coupled electrons and holes. Conversely, the pseudo-spin magnetization quantum transport equations incorporate the pseudo-spin/real-spin coupling, as well as the pseudo-spin coupling with the total charge in the system. The SMQTEs are summarily cast in more physically transparent forms in Sec. VIII

## II. NONEQUILIBRIUM QUANTUM SUPERFIELD THEORY

Our starting point is the general quantum transport expressions for fermions as obtained from the real-time quantum superfield theoretical formulation of Buot<sup>5,7</sup>:

$$\begin{aligned}
i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^{\lessgtr} &= [vG^{\lessgtr} - G^{\lessgtr}v^T] \\
&+ [\Sigma^r G^{\lessgtr} - G^{\lessgtr}\Sigma^a] + [\Sigma^{\lessgtr}G^a - G^r\Sigma^{\lessgtr}] \\
&+ [\Delta_{hh}^r g_{ee}^{\lessgtr} - g_{hh}^{\lessgtr}\Delta_{ee}^a] \\
&+ [\Delta_{hh}^{\lessgtr} g_{ee}^a - g_{hh}^r\Delta_{ee}^{\lessgtr}].
\end{aligned} \tag{3}$$

The last two brackets account for the Cooper pairings between Fermions of the same specie. These do not concern us in this paper (their corresponding transport equations<sup>5,7</sup> are important in nonequilibrium superconductivity). In what follows we will drop these last two brackets of the RHS of Eq. (3).<sup>25</sup>

### III. MULTI-BAND QUANTUM TRANSPORT EQUATIONS

In the absence of pairing between fermions of the same-specie Eq. (3) becomes, by explicitly writing the quantum arguments, as

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{\alpha\beta}^{\leqslant}(12) \\
& = [v_{\alpha}\delta_{\alpha\gamma}\delta_{1\bar{2}}G_{\gamma\beta}^{\leqslant}(\bar{2}2) - G_{\alpha\gamma}^{\leqslant}(1\bar{2})v_{\gamma}^T\delta_{\gamma\beta}\delta_{\bar{2}2}] \\
& + [\Sigma_{\alpha\gamma}^r(1\bar{2})G_{\gamma\beta}^{\leqslant}(\bar{2}2) - G_{\alpha\gamma}^{\leqslant}(1\bar{2})\Sigma_{\gamma\beta}^a(\bar{2}2)] \\
& + [\Sigma_{\alpha\gamma}^{\leqslant}(1\bar{2})G_{\gamma\beta}^a(\bar{2}2) - G_{\alpha\gamma}^r(1\bar{2})\Sigma_{\gamma\beta}^{\leqslant}(\bar{2}2)], \tag{4}
\end{aligned}$$

where the Greek subscript indices correspond to discrete band indices, and the numeral indices correspond to the two-point space-time arguments. In what follows we will treat the two-band model of a semiconductor and replace by  $v$  and  $c$ , the Greek indices for the valence and conduction band quantum labels, respectively.

#### A. Electron-Hole Picture

The transport equation for the conduction-band electrons in the electron-hole or defect representation reduces to

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{cc}^{e-h,<}(12) \\
& = [v_c(1\xi)G_{cc}^{<}(\xi 2) - G_{cc}^{<}(1\xi)v_c^T(\xi 2)] \\
& + [\Sigma_{cc}^r(1\xi)G_{cc}^{<}(\xi 2) - G_{cc}^{<}(1\xi)\Sigma_{cc}^a(\xi 2)] \\
& + [\Sigma_{cc}^{<}(1\xi)G_{cc}^a(\xi 2) - G_{cc}^r(1\xi)\Sigma_{cc}^{<}(\xi 2)] \\
& + [\Delta_{hh,cv}^{e-h,r}(1\xi)g_{ee,vc}^{e-h,<}(\xi 2) - g_{hh,cv}^{e-h,<}(1\xi)\Delta_{ee,vc}^{e-h,a}(\xi 2)] \\
& + [\Delta_{hh,cv}^{e-h,<}(1\xi)g_{ee,vc}^{e-h,a}(\xi 2) - g_{hh,cv}^{e-h,r}(1\xi)\Delta_{ee,vc}^{e-h,<}(\xi 2)], \tag{5}
\end{aligned}$$

where the off-diagonal self-energies,  $\Sigma_{\alpha\beta}$ , and Green's functions,  $G_{\alpha\beta}$  are denoted by  $\Delta^{e-h}$  and  $g^{e-h}$ , respectively, in analogy to the  $\Delta$ -function and anomalous Green's function,  $\mathcal{F}$ , i.e.,  $g^{e-h} \Rightarrow \mathcal{F}$  in the theory of superconductivity.<sup>26–28</sup> In Eq. (5), there is no Cooper pairing between electrons; moreover, the electron and electron-hole pictures coincides for the conduction band.

To obtain the equation for the hole density from the general superfield formulation for electrons, we make use of the electron-hole conversion table given in Ref.<sup>6,7</sup>, where the relevant portion is reproduced below,

Table 1. Mapping from Electron to Electron-Hole Picture

<b>electron picture</b>	$\langle e - \text{field} \rangle$	$\langle e - h \text{ field} \rangle$	<b>e - h picture</b>
$-i\hbar G_{vv}^<(12)$	$\langle \psi_v^\dagger(2) \psi_v(1) \rangle$	$\langle \phi_v(2) \phi_v^\dagger(1) \rangle$	$i\hbar G_{vv}^{h,>T}(12)$
$-i\hbar G_{vc}^<(12)$	$\langle \psi_c^\dagger(2) \psi_v(1) \rangle$	$\langle \psi_c^\dagger(2) \phi_v^\dagger(1) \rangle$	$-i\hbar g_{ee,vc}^{e-h,<}(12)$
$-i\hbar G_{cv}^<(12)$	$\langle \psi_v^\dagger(2) \psi_c(1) \rangle$	$\langle \phi_v(2) \psi_c(1) \rangle$	$-i\hbar g_{hh,cv}^{e-h,<}(12)$
$-i\hbar G_{cc}^<(12)$	$\langle \psi_c^\dagger(2) \psi_c(1) \rangle$	$\langle \psi_c^\dagger(2) \psi_c(1) \rangle$	$-i\hbar G_{cc}^{e-h,<}(12)$
$i\hbar G_{vv}^{>T}(12)$	$\langle \psi_v(2) \psi_v^\dagger(1) \rangle$	$\langle \phi_v^\dagger(2) \phi_v(1) \rangle$	$-i\hbar G_{vv}^{h,<}(12)$
$i\hbar G_{vv}^{>}(12)$	$\langle \psi_v(1) \psi_v^\dagger(2) \rangle$	$\langle \phi_v^\dagger(1) \phi_v(2) \rangle$	$-i\hbar G_{vv}^{h,<T}(12)$

We also have the following Tables, which can also be similarly applied to the self-energies,

Table 2.

<b>electron picture</b>	<b>e - h picture</b>
$G_{vv}^r(12)$	$-G_{vv}^{e-h,aT}(12)$
$G_{vc}^r(12)$	$g_{ee,vc}^{e-h,r}(12)$
$G_{cv}^r(12)$	$g_{hh,cv}^{e-h,r}(12)$
$G_{cc}^r(12)$	$G_{cc}^{e-h,r}(12)$

Table 3.

<b>electron picture</b>	<b>e - h picture</b>
$G_{vv}^a(12)$	$-G_{vv}^{e-h,rT}(12)$
$G_{vc}^a(12)$	$g_{ee,vc}^{e-h,a}(12)$
$G_{cv}^a(12)$	$g_{hh,cv}^{e-h,a}(12)$
$G_{cc}^a(12)$	$G_{cc}^{e-h,a}(12)$

From the first table, we see that the hole density,  $-i\hbar G_{vv}^{e-h,<}(12)$  correspond to  $i\hbar G_{vv}^{>T}(12)$  in the electron energy-band representation. Thus, to obtain the equation for the hole density given by  $-i\hbar G_{vv}^{e-h,<}(12)$ , one may look for the equation for  $i\hbar G_{vv}^{>T}(12)$  in the general nonequilibrium formulation of electrons. The equation for  $i\hbar G_{vv}^{>T}(12)$  can be obtained from the equation for  $G_{vv}^{>}$ , by taking the complex conjugate of the above equation and making use of the relation

$$G^{>\dagger}(12) = -G^{>T}(12). \quad (6)$$

We have the transport equation for  $G^>(12)$  given by

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{vv}^>(12) \\
&= [v_{vv}^T(1\xi) G_{vv}^>(\xi 2) - G_{vv}^>(1\xi) v_{vv}(\xi 2)] \\
&+ [\Sigma_{vv}^r(1\xi) G_{vv}^>(\xi 2) - G_{vv}^>(1\xi) \Sigma_{vv}^a(\xi 2)] \\
&+ [\Sigma_{vc}^r(1\xi) G_{cv}^>(\xi 2) - G_{vc}^>(1\xi) \Sigma_{cv}^a(\xi 2)] \\
&+ [\Sigma_{vv}^>(1\xi) G_{vv}^a(\xi 2) - G_{vv}^r(1\xi) \Sigma_{vv}^>(\xi 2)] \\
&+ [\Sigma_{vc}^>(1\xi) G_{cv}^a(\xi 2) - G_{vc}^r(1\xi) \Sigma_{cv}^>(\xi 2)]
\end{aligned} \tag{7}$$

Taking the complex conjugate of Eq. (7), we obtain

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{vv}^{>T}(12) \\
&= [-G_{vv}^>(2\xi) v_{vv}^T(\xi 1) + v_{vv}^T(\xi 2) G_{vv}^{>T}(1\xi)] \\
&+ [-G_{vv}^>(2\xi) \Sigma_{vv}^a(\xi 1) + \Sigma_{vv}^r(2\xi) G_{vv}^>(\xi 1)] \\
&+ [-G_{cv}^>(2\xi) \Sigma_{vc}^a(\xi 1) + \Sigma_{cv}^r(2\xi) G_{vc}^>(\xi 1)] \\
&+ [-G_{vv}^r(2\xi) \Sigma_{vv}^>(\xi 1) + \Sigma_{vv}^>(2\xi) G_{vv}^a(\xi 1)] \\
&+ [-G_{cv}^r(2\xi) \Sigma_{vc}^>(\xi 1) + \Sigma_{cv}^>(2\xi) G_{vc}^a(\xi 1)].
\end{aligned} \tag{8}$$

Applying the relation

$$-G_{vv}^{>T}(12) = -G_{vv}^>(21) = G_{vv}^{e-h,<}(12),$$

and going entirely to the defect representation for the rest of the terms, we obtain for the time evolution of the hole density without Cooper pairing between holes.

$$\begin{aligned}
& -i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{vv}^{e-h,<}(12) \\
&= -[-v_{vv}(1\xi) G_{vv}^{e-h,<}(\xi 2) + G_{vv}^{e-h,<}(1\xi) v_{vv}^T(\xi 2)] \\
&- [-\Sigma_{vv}^{e-h,r}(1\xi) G_{vv}^{e-h,<}(\xi 2) + G_{vv}^{e-h,<}(1\xi) \Sigma_{vv}^{e-h,a}(\xi 2)] \\
&- [-\Sigma_{vc}^{e-h,r}(1\xi) G_{cv}^{e-h,<}(\xi 2) + G_{vc}^{e-h,<}(1\xi) \Sigma_{cv}^{e-h,a}(\xi 2)] \\
&- [-\Sigma_{vv}^{e-h,<}(1\xi) G_{vv}^{e-h,a}(\xi 2) + G_{vv}^{e-h,r}(1\xi) \Sigma_{vv}^{e-h,<}(\xi 2)] \\
&- [-\Sigma_{vc}^{e-h,<}(1\xi) G_{cv}^{e-h,a}(\xi 2) + G_{vc}^{e-h,r}(1\xi) \Sigma_{cv}^{e-h,<}(\xi 2)].
\end{aligned} \tag{9}$$

Replacing the off-diagonal self-energies,  $\Sigma_{\alpha\beta}$ , and Green's functions,  $G_{\alpha\beta}$  by  $\Delta^{e-h}$  and  $g^{e-h}$ ,

respectively, as was done in Eq. (5), allow us to rewrite the equation as

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{vv}^{e-h,<} (12) \\
&= - \left[ v_{vv} (1\xi) G_{vv}^{e-h,<} (\xi 2) - G_{vv}^{e-h,<} (1\xi) v_{vv}^T (\xi 2) \right] \\
&- \left[ \Sigma_{vv}^{e-h,r} (1\xi) G_{vv}^{e-h,<} (\xi 2) - G_{vv}^{e-h,<} (1\xi) \Sigma_{vv}^{e-h,a} (\xi 2) \right] \\
&- \left[ \Delta_{ee,vc}^{e-h,r} (1\xi) g_{hh,cv}^{e-h,<} (\xi 2) - g_{ee,vc}^{e-h,<} (1\xi) \Delta_{hh,cv}^{e-h,a} (\xi 2) \right] \\
&- \left[ \Sigma_{vv}^{e-h,<} (1\xi) G_{vv}^{e-h,a} (\xi 2) - G_{vv}^{e-h,r} (1\xi) \Sigma_{vv}^{e-h,<} (\xi 2) \right] \\
&- \left[ \Delta_{ee,vc}^{e-h,<} (1\xi) g_{hh,cv}^{e-h,a} (\xi 2) - g_{ee,vc}^{e-h,r} (1\xi) \Delta_{cv}^{e-h,<} (\xi 2) \right]. \tag{10}
\end{aligned}$$

Compared with Eq. (5) for the conduction band, one can readily see that for flat bands, or atomic limit, the residual terms in these two equations are equal,

$$\begin{aligned}
i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{vv}^{e-h,<} (12) &\Rightarrow - \left[ \Delta_{ee,vc}^{e-h,r} g_{hh,cv}^{e-h,<} - g_{ee,vc}^{e-h,<} \Delta_{hh,cv}^{e-h,a} \right] \\
&= \left[ \Delta_{hh,cv}^{e-h,r} g_{ee,vc}^{e-h,<} - g_{hh,cv}^{e-h,<} \Delta_{ee,vc}^{e-h,a} \right], \\
i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{cc}^{e-h,<} (12) &\Rightarrow \left[ \Delta_{hh,cv}^{e-h,r} g_{ee,vc}^{e-h,<} - g_{hh,cv}^{e-h,<} \Delta_{ee,vc}^{e-h,a} \right], \tag{11}
\end{aligned}$$

since  $\Delta_{ee,vc}^{e-h,r} = \Delta_{ee,vc}^{e-h,a}$ , and  $\Delta_{hh,cv}^{e-h,a} = \Delta_{hh,cv}^{e-h,r}$ . This equality simply states that the rate of change of the holes in the valence band is the same as the rate of change of the electrons in the conduction band, or the rate of electron creation in conduction band equals the rate of hole creation in the valence band, and *vice versa*.

If we use the electron picture to write the transport equation for the valence band, the resulting equation would be similar to Eq. (10) without the superscript  $e-h$  and with the right hand side of that equation multiplied by  $-1$ . Then one can also easily see that for flat-band case, the rate of change of electrons in the conduction band and that of the valence band would have opposite sign, i.e., the rate of creation of electrons in the conduction band occurs at the expense of equal rate of loss of electrons in the valence band. Later in this paper, we will use the electron picture to investigate the pseudo-spin transport equations. There, the total charge represents the total net charge to be used in the Poisson equation. Of course the total net charge at each point of the phase-space is the charge represented by the total number of electrons in the conduction and valence band minus the total positive background charge of the crystal lattice.

Note that the single particle Hamiltonian  $v_{vv}$  is not really a two-point function, unlike



the correlation functions. The above equation reduces to

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G_{vv}^{e-h,<} (12) \\
&= - \left[ v_{vv} (1\xi) G_{vv}^{e-h,<} (\xi 2) - G_{vv}^{e-h,<} (1\xi) v_{vv}^T (\xi 2) \right] \\
&+ \left[ \Sigma_{vv}^{e-h,r} (1\xi) G_{vv}^{e-h,<} (\xi 2) - G_{vv}^{e-h,<} (1\xi) \Sigma_{vv}^{e-h,a} (\xi 2) \right] \\
&+ \left[ \Sigma_{vv}^{e-h,<} (1\xi) G_{vv}^{e-h,a} (\xi 2) - G_{vv}^{e-h,r} (1\xi) \Sigma_{vv}^{e-h,<} (\xi 2) \right] \\
&+ \left[ g_{ee,vc}^{e-h,>} (2\xi) \Delta_{hh,cv}^{e-h,a} (\xi 1) - \Delta_{ee,vc}^{e-h,r} (2\xi) g_{hh,cv}^{e-h,>} (\xi 1) \right] \\
&+ \left[ g_{ee,vc}^{e-h,r} (2\xi) \Delta_{hh,cv}^{e-h,>} (\xi 1) - \Delta_{ee,vc}^{e-h,>} (2\xi) g_{hh,cv}^{e-h,a} (\xi 1) \right]. \tag{12}
\end{aligned}$$

The equations for the interband or polarization terms,  $g_{hh}^{e-h,<}$ ,  $g_{ee}^{e-h,<}$ , etc. can be obtained from the equations for  $G_{\alpha\beta}^{<}$  (12), where  $\alpha \neq \beta$ . In the electron-hole or defect representation, the equation for  $g_{hh,cv}^{e-h,<}$  is determined from the general superfield formalism given by the equation for  $G_{cv}^{<}$  (12). Upon transforming to the electron-hole picture, the change of polarization due to the destruction of electron-hole pairs brought about by Auger recombination, optical de-excitation, and other recombination processes is given in the defect representation as,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) g_{hh,cv}^{e-h,<} (12) \\
&= \left[ v_{cc} (\xi 2) g_{hh,cv}^{e-h,<} (1\xi) - g_{hh,cv}^{e-h,<} (\xi 2) v_{vv}^T (1\xi) \right] \\
&+ \left[ \Sigma_{cc}^{e-h,r} (1\xi) g_{hh,cv}^{e-h,<} (\xi 2) - G_{cc}^{e-h,<} (1\xi) \Delta_{hh,cv}^{e-h,a} (\xi 2) \right] \\
&+ \left[ \Sigma_{cc}^{e-h,<} (1\xi) g_{hh,cv}^{e-h,a} (\xi 2) - G_{cc}^{e-h,r} (1\xi) \Delta_{hh,cv}^{e-h,<} (\xi 2) \right] \\
&+ \left[ -\Delta_{hh,cv}^{e-h,r} (1\xi) G_{vv}^{e-h,>T} (\xi 2) + g_{hh,cv}^{e-h,<} (1\xi) \Sigma_{vv}^{e-h,rT} (\xi 2) \right] \\
&+ \left[ -\Delta_{hh,cv}^{e-h,<} (1\xi) G_{vv}^{e-h,rT} (\xi 2) + g_{hh,cv}^{e-h,r} (1\xi) \Sigma_{vv}^{e-h,>T} (\xi 2) \right]. \tag{13}
\end{aligned}$$

Similarly, upon going over to the defect representation of the reverse process, i.e., the change of polarization due to the creation of electron-hole pairs by impact ionization, optical

excitation, Zener tunneling, and other excitation processes is therefore given by

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) g_{ee,vc}^{e-h,<}(\xi 2) \\
&= [v_{vv}(1\xi) g_{ee,vc}^{e-h,<}(\xi 2) - g_{ee,vc}^{e-h,<}(1\xi) v_c^T(\xi 2)] \\
&+ [-\Sigma_{vv}^{e-h,aT}(1\xi) g_{ee,vc}^{e-h,<}(\xi 2) + G_{vv}^{e-h,>T}(1\xi) \Delta_{ee,vc}^{e-h,a}(\xi 2)] \\
&+ [-\Sigma_{vv}^{e-h,>T}(1\xi) g_{ee,vc}^{e-h,a}(\xi 2) + G_{vv}^{e-h,aT}(1\xi) \Delta_{ee,vc}^{e-h,<}(\xi 2)] \\
&+ [\Delta_{ee,vc}^{e-h,r}(1\xi) G_{cc}^{e-h,<}(\xi 2) - g_{ee,vc}^{e-h,<}(1\xi) \Sigma_{cc}^{e-h,a}(\xi 2)] \\
&+ [\Delta_{ee,vc}^{e-h,<}(1\xi) G_{cc}^{e-h,a}(\xi 2) - g_{ee,vc}^{e-h,r}(1\xi) \Sigma_{cc}^{e-h,<}(\xi 2)]. \tag{14}
\end{aligned}$$

Note the presence of the 'transposed' terms, namely,  $G_{vv}^{e-h,>T}(\xi 2)$ ,  $\Sigma_{vv}^{e-h,rT}(\xi 2)$ ,  $G_{vv}^{e-h,rT}(\xi 2)$ , and  $\Sigma_{vv}^{e-h,>T}(\xi 2)$  in Eq. (13) and  $\Sigma_{vv}^{e-h,aT}(1\xi)$ ,  $G_{vv}^{e-h,aT}(1\xi)$ ,  $G_{vv}^{e-h,>T}(1\xi)$ , and  $\Sigma_{vv}^{e-h,>T}(1\xi)$  in Eq. (14) of the interband 'pairing' correlation functions.

In the calculation of the *pseudo-spin* transport equations the transposed quantities in Eqs. (13) and (14) do not enter since we use the electron picture to provide the two states for the electrons, namely the two quantum labels  $c$  and  $v$  in order to derive the pseudo-spin quantum transport equations. Since there is a one-to-one mapping between the *electron picture* and *electron-hole picture* except for the valence band as shown in the tables above, all one does to go from electron-hole picture to the electron picture is to change the sign of the right-hand side of Eq. (12) and substitute for the transposed quantities their equivalent expressions in the electron picture.

#### IV. EQUATIONS FOR BLOCH ELECTRONS WITH SPIN

In the presence of Pauli-Dirac spin degree of freedom, Eqs. (5), (12), (13), and (14) become matrix equations. These are given in the Appendix.

##### A. Spin Canonical $2 \times 2$ Matrix Forms

In spintronics, we are interested in the time-dependent evolution of the multi-band polarization and magnetization densities,  $S_{\alpha\beta,z} = i\hbar (G_{\alpha\beta,\uparrow\uparrow}^{<} - G_{\alpha\beta,\downarrow\downarrow}^{<})$ , as these are transported across the device. This leads us to transform all  $2 \times 2$  matrices into their spin canonical forms, defined here as expansion in terms of the Pauli spin matrices and identity.

First let us partition the *total* nonequilibrium Green's functions (TNEGF) into  $2 \times 2$  submatrix components, by virtue of the spin indices, as

$$\begin{pmatrix} G_{cc,\uparrow\uparrow}^{e-h,<}(12) & G_{cc,\uparrow\downarrow}^{e-h,<}(12) & g_{hh,cv,\uparrow\uparrow}^{e-h,<}(12) & g_{hh,cv,\uparrow\downarrow}^{e-h,<}(12) \\ G_{cc,\downarrow\uparrow}^{e-h,<}(12) & G_{cc,\downarrow\downarrow}^{e-h,<}(12) & g_{hh,cv,\downarrow\uparrow}^{e-h,<}(12) & g_{hh,cv,\downarrow\downarrow}^{e-h,<}(12) \\ g_{ee,vc,\downarrow\downarrow}^{e-h,<}(12) & g_{ee,vc,\downarrow\uparrow}^{e-h,<}(12) & G_{vv,\uparrow\uparrow}^{e-h,<}(12) & G_{vv,\uparrow\downarrow}^{e-h,<}(12) \\ g_{ee,vc,\uparrow\downarrow}^{e-h,<}(12) & g_{ee,vc,\uparrow\uparrow}^{e-h,<}(12) & G_{vv,\downarrow\uparrow}^{e-h,<}(12) & G_{vv,\downarrow\downarrow}^{e-h,<}(12) \end{pmatrix} = \begin{pmatrix} (CC) & (CV) \\ (VC) & (VV) \end{pmatrix}, \quad (15)$$

where  $(CC)$ ,  $(CV)$ ,  $(VC)$ , and  $(VV)$  are  $2 \times 2$  submatrices. We can form four coupled spin magnetization quantum transport equations for each submatrix, giving us 16 coupled magnetization quantum transport equations.

Thus, we first transform the  $2 \times 2$  matrix variables in Eq. (15) into their spin-canonical forms using the Pauli-matrices,

$$(CC) = \frac{1}{2} \left( S_{cc,o} I + \vec{S}_{cc} \cdot \vec{\sigma} \right), \quad (16)$$

$$(CV) = \frac{1}{2} \left( S_{cv,o} I + \vec{S}_{cv} \cdot \vec{\sigma} \right), \quad (17)$$

$$(VC) = \frac{1}{2} \left( S_{vc,o} I + \vec{S}_{vc} \cdot \vec{\sigma} \right), \quad (18)$$

$$(VV) = \frac{1}{2} \left( S_{vv,o} I + \vec{S}_{vv} \cdot \vec{\sigma} \right), \quad (19)$$

where we dropped the superscript  $<$  in the above spin-canonical form, this is to be understood in what follows unless otherwise specifically specified. The spin canonical expansion in terms of the Pauli matrices basically separates the spin-independent terms from the spin-dependent terms.

In the case of the conduction, Eq. (16), and valence bands, Eq.(19), the coefficients of the identity matrix, namely,  $S_{cc,o}$  and  $S_{vv,o}$  represent the charge density of electrons in the conduction band and charge density of holes in the valence band, respectively. Similarly,  $S_{cv,o}$  and  $S_{vc,o}$  represent the probability density for the annihilation and creation of electron-hole pairs, respectively. The coefficients of the vector,  $\vec{\sigma}$ , represent the corresponding spin magnetization vectors. Thus the canonical form basically performs the essential function of separating particle charge and spin.

We will later show that the coefficients of the identity matrix given above can also be expressed in terms of the pseudo-spin vector and the total charge in our multi-band system.

## B. Intraband Nonequilibrium Spin Correlation Functions

To establish the notations used in this paper, we give the spin canonical forms for the single particle Hamiltonian, various correlation functions, and self-energies.

For the intraband canonical terms, we have for the intraband spin-*subtraces* scalar correlation functions,

$$S_{cc,o}^{r,a,<} = \left( G_{cc,\uparrow\uparrow}^{e-h,r,a,<} + G_{cc,\downarrow\downarrow}^{e-h,r,a,<} \right), \quad (20)$$

$$S_{vv,o}^{r,a,<} = \left( G_{vv,\uparrow\uparrow}^{e-h,r,a,<} + G_{vv,\downarrow\downarrow}^{e-h,r,a,<} \right). \quad (21)$$

The intraband spin-correlation vector components for electrons are,

$$S_{cc,x}^{r,a,<} = \left( G_{cc,\downarrow\uparrow}^{e-h,r,a,<} + G_{cc,\uparrow\downarrow}^{e-h,r,a,<} \right),$$

$$iS_{cc,y}^{r,a,<} = \left( G_{cc,\downarrow\uparrow}^{e-h,r,a,<} - G_{cc,\uparrow\downarrow}^{e-h,r,a,<} \right),$$

$$S_{cc,z}^{r,a,<} = \left( G_{cc,\uparrow\uparrow}^{e-h,r,a,<} - G_{cc,\downarrow\downarrow}^{e-h,r,a,<} \right),$$

and for holes,

$$S_{vv,x}^{r,a,<} = \left( G_{vv,\downarrow\uparrow}^{e-h,r,a,<} + G_{vv,\uparrow\downarrow}^{e-h,r,a,<} \right),$$

$$iS_{vv,y}^{r,a,<} = \left( G_{vv,\downarrow\uparrow}^{e-h,r,a,<} - G_{vv,\uparrow\downarrow}^{e-h,r,a,<} \right),$$

$$S_{vv,z}^{r,a,<} = \left( G_{vv,\uparrow\uparrow}^{e-h,r,a,<} - G_{vv,\downarrow\downarrow}^{e-h,r,a,<} \right).$$

## C. Interband Nonequilibrium Spin Correlation Functions

For the interband canonical terms, we have interband spin-*subtraces* scalar correlation functions,

$$S_{cv,o}^{r,a,<} = \left( g_{hh,cv,\uparrow\uparrow}^{e-h,r,a,<} + g_{hh,cv,\downarrow\downarrow}^{e-h,r,a,<} \right), \quad (22)$$

$$S_{vc,o}^{r,a,<} = \left( g_{ee,vc,\downarrow\downarrow}^{e-h,r,a,<} + g_{ee,vc,\uparrow\uparrow}^{e-h,r,a,<} \right), \quad (23)$$

and for the interband spin-vector correlation components,

$$S_{cv,x}^{r,a,<} = \left( g_{hh,cv,\downarrow\uparrow}^{e-h,r,a,<} + g_{hh,cv,\uparrow\downarrow}^{e-h,r,a,<} \right),$$

$$iS_{cv,y}^{r,a,<} = \left( g_{hh,cv,\downarrow\uparrow}^{e-h,r,a,<} - g_{hh,cv,\uparrow\downarrow}^{e-h,r,a,<} \right),$$

$$S_{cv,z}^{r,a,<} = \left( g_{hh,cv,\uparrow\uparrow}^{e-h,r,a,<} - g_{hh,cv,\downarrow\downarrow}^{e-h,r,a,<} \right).$$

The corresponding interband reverse process 'conjugate' correlations are,

$$\begin{aligned} S_{vc,x}^{r,a,<} &= \left( g_{ee,vc,\uparrow\downarrow}^{e-h,r,a,<} + g_{ee,vc,\downarrow\uparrow}^{e-h,r,a,<} \right), \\ iS_{vc,y}^{r,a,<} &= \left( g_{ee,vc,\uparrow\downarrow}^{e-h,r,a,<} - g_{ee,vc,\downarrow\uparrow}^{e-h,r,a,<} \right), \\ S_{vc,z}^{r,a,<} &= \left( g_{ee,vc,\downarrow\downarrow}^{e-h,r,a,<} - g_{ee,vc,\uparrow\uparrow}^{e-h,r,a,<} \right). \end{aligned}$$

#### D. Single-Particle Hamiltonian Spin Canonical Forms

We have for the single-particle Hamiltonian for the electrons expressed in canonical forms, assuming the presence of external magnetic field and/or spin-orbit coupling,

$$v_{c,\sigma\sigma'} = \frac{1}{2} \left( \bar{H}_c \hat{I} + \vec{\mathcal{B}}_c \cdot \vec{\sigma} \right),$$

where the upper bar in  $\bar{H}_c$  indicates the subtrace of the  $2 \times 2$  spin matrix for the conduction band. Similarly, for the holes we have,

$$v_{v,\sigma\sigma'} = \frac{1}{2} \left( \bar{H}_v \hat{I} + \vec{\mathcal{B}}_v \cdot \vec{\sigma} \right).$$

These lead to the matrix expression for electrons,

$$v_{c,\sigma\sigma'} = \frac{1}{2} \begin{pmatrix} \bar{H}_c + \mathcal{B}_{c,z} & \mathcal{B}_{c,x} - i\mathcal{B}_{c,y} \\ \mathcal{B}_{c,x} + i\mathcal{B}_{c,y} & \bar{H}_c - \mathcal{B}_{c,z} \end{pmatrix},$$

and for holes,

$$v_{v,\sigma\sigma'} = \frac{1}{2} \begin{pmatrix} \bar{H}_v + \mathcal{B}_{v,z} & \mathcal{B}_{v,x} - i\mathcal{B}_{v,y} \\ \mathcal{B}_{v,x} + i\mathcal{B}_{v,y} & \bar{H}_v - \mathcal{B}_{v,z} \end{pmatrix}.$$

The components of  $\vec{\mathcal{B}}_c$  and  $\vec{\mathcal{B}}_v$  are real valued, since the single-particle Hamiltonian,  $v$ , is Hermitian. These are the effective magnetic field components, multiplied by  $\mu_B g$ , which accounts for the external magnetic fields as well as the effects of spin-orbit coupling.

#### E. Canonical Forms for the Spin-Dependent Electrons Self-Energy

Similarly, we have for the spin-canonical forms for the electron self-energies,

$$\Sigma_{cc,\sigma\sigma'}^r = \frac{1}{2} \begin{pmatrix} \bar{\Sigma}_{cc}^r + \Xi_{cc,z}^r & \Xi_{cc,x}^r - i\Xi_{cc,y}^r \\ \Xi_{cc,x}^r + i\Xi_{cc,y}^r & \bar{\Sigma}_{cc}^r - \Xi_{cc,z}^r \end{pmatrix},$$

$$\Sigma_{cc,\sigma\sigma'}^a = \frac{1}{2} \begin{pmatrix} \bar{\Sigma}_{cc}^a + \Xi_{cc,z}^a & \Xi_{cc,x}^a - i\Xi_{cc,y}^a \\ \Xi_{cc,x}^a + i\Xi_{cc,y}^a & \bar{\Sigma}_{cc}^a - \Xi_{cc,z}^a \end{pmatrix},$$

$$\Sigma_{cc,\sigma\sigma'}^< = \frac{1}{2} \begin{pmatrix} \bar{\Sigma}_{cc}^< + \Xi_{cc,z}^< & \Xi_{cc,x}^< - i\Xi_{cc,y}^< \\ \Xi_{cc,x}^< + i\Xi_{cc,y}^< & \bar{\Sigma}_{cc}^< - \Xi_{cc,z}^< \end{pmatrix}.$$

### F. Spin-Canonical Forms for the Holes Self-Energy

$$\Sigma_{vv,\sigma\sigma'}^r = \frac{1}{2} \begin{pmatrix} \bar{\Sigma}_{vv}^r + \Xi_{vv,z}^r & \Xi_{vv,x}^r - i\Xi_{vv,y}^r \\ \Xi_{vv,x}^r + i\Xi_{vv,y}^r & \bar{\Sigma}_{vv}^r - \Xi_{vv,z}^r \end{pmatrix},$$

$$\Sigma_{vv,\sigma\sigma'}^a = \frac{1}{2} \begin{pmatrix} \bar{\Sigma}_{vv}^a + \Xi_{vv,z}^a & \Xi_{vv,x}^a - i\Xi_{vv,y}^a \\ \Xi_{vv,x}^a + i\Xi_{vv,y}^a & \bar{\Sigma}_{vv}^a - \Xi_{vv,z}^a \end{pmatrix},$$

$$\Sigma_{vv,\sigma\sigma'}^< = \frac{1}{2} \begin{pmatrix} \bar{\Sigma}_{vv}^< + \Xi_{vv,z}^< & \Xi_{vv,x}^< - i\Xi_{vv,y}^< \\ \Xi_{vv,x}^< + i\Xi_{vv,y}^< & \bar{\Sigma}_{vv}^< - \Xi_{vv,z}^< \end{pmatrix}.$$

### G. Canonical Forms of Electron-Hole Pairing Self-Energy

$$\Delta_{hh,cv,\sigma\sigma'}^{e-h,r} = \frac{1}{2} \left( \bar{\Delta}_{hh,cv}^r + \vec{\delta}_{hh,cv}^r \cdot \vec{\sigma} \right),$$

$$\Delta_{hh,cv,\sigma\sigma'}^{e-h,<} = \frac{1}{2} \left( \bar{\Delta}_{hh,cv}^< + \vec{\delta}_{hh,cv}^< \cdot \vec{\sigma} \right),$$

$$\Delta_{ee,vc,\sigma\sigma'}^{e-h,a} = \frac{1}{2} \left( \bar{\Delta}_{ee,vc}^a + \vec{\delta}_{ee,vc}^a \cdot \vec{\sigma} \right),$$

$$\Delta_{ee,vc,\sigma\uparrow}^{e-h,<} = \frac{1}{2} \left( \bar{\Delta}_{ee,vc}^< + \vec{\delta}_{ee,vc}^< \cdot \vec{\sigma} \right),$$

$$\Delta_{hh,cv,\sigma\sigma'}^{e-h,r} = \frac{1}{2} \begin{pmatrix} \bar{\Delta}_{hh,cv}^r + \delta_{hh,cv,z}^r & \delta_{hh,cv,x}^r - i\delta_{hh,cv,y}^r \\ \delta_{hh,cv,x}^r + i\delta_{hh,cv,y}^r & \bar{\Delta}_{hh,cv}^r - \delta_{hh,cv,z}^r \end{pmatrix},$$

$$\Delta_{hh,cv,\sigma\sigma'}^{e-h,<} = \frac{1}{2} \begin{pmatrix} \bar{\Delta}_{hh,cv}^< + \delta_{hh,cv,z}^< & \delta_{hh,cv,x}^< - i\delta_{hh,cv,y}^< \\ \delta_{hh,cv,x}^< + i\delta_{hh,cv,y}^< & \bar{\Delta}_{hh,cv}^< - \delta_{hh,cv,z}^< \end{pmatrix},$$

$$\Delta_{ee,vc,\sigma\sigma'}^{e-h,a} = \frac{1}{2} \begin{pmatrix} \bar{\Delta}_{ee,vc}^a + \delta_{ee,vc,z}^a & \delta_{ee,vc,x}^a - i\delta_{ee,vc,y}^a \\ \delta_{ee,vc,x}^a + i\delta_{ee,vc,y}^a & \bar{\Delta}_{ee,vc}^a - \delta_{ee,vc,z}^a \end{pmatrix},$$

$$\Delta_{ee,vc,\sigma\sigma'}^{e-h,<} = \frac{1}{2} \begin{pmatrix} \bar{\Delta}_{ee,vc}^{<} + \delta_{ee,vc,z}^{<} & \delta_{ee,vc,x}^{<} - i\delta_{ee,vc,y}^{<} \\ \delta_{ee,vc,x}^{<} + i\delta_{ee,vc,y}^{<} & \bar{\Delta}_{ee,vc}^{<} - \delta_{ee,vc,z}^{<} \end{pmatrix}.$$

## V. NONEQUILIBRIUM PAULI-DIRAC SPIN EQUATIONS

The twelve transport equations for the components of the multi-band spin magnetization transport equations can be written in more compact form as spin-vector equations, a generalization of Eq. (1) to multi-band spin-magnetization quantum transport equations.

The results can be summarized as four-coupled *vector* equations for the spin magnetization-distribution functions,

$$\begin{aligned} & i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{cc}^{<} \\ &= \frac{1}{2} \left\{ \left[ \bar{H}_c, \vec{S}_{cc}^{<} \right] + \left[ \vec{\mathcal{B}}_{cc}, S_{cc,o}^{<} \right] + i \left[ \vec{\mathcal{B}}_{cc} \times \vec{S}_{cc}^{<} - \vec{S}_{cc}^{<} \times \vec{\mathcal{B}}_{cc} \right] \right\} \\ &+ \frac{1}{2} \left[ \begin{aligned} & + \left[ \bar{\Sigma}_{cc}^r \vec{S}_{cc}^{<} - \vec{S}_{cc}^{<} \bar{\Sigma}_{cc}^a \right] + \left[ \vec{\Xi}_{cc}^r S_{cc,o}^{<} - S_{cc,o}^{<} \vec{\Xi}_{cc}^a \right] \\ & + i \left[ \vec{\Xi}_{cc}^r \times \vec{S}_{cc}^{<} - \vec{S}_{cc}^{<} \times \vec{\Xi}_{cc}^a \right] \end{aligned} \right] \\ &+ \frac{1}{2} \left[ \begin{aligned} & \left[ \bar{\Sigma}_{cc}^{<} \vec{S}_{cc}^a - \vec{S}_{cc}^r \bar{\Sigma}_{cc}^{<} \right] + \left[ \vec{\Xi}_{cc}^{<} S_{cc,o}^a - S_{cc,o}^r \vec{\Xi}_{cc}^{<} \right] \\ & + i \left[ \vec{\Xi}_{cc}^{<} \times \vec{S}_{cc}^a - \vec{S}_{cc}^r \times \vec{\Xi}_{cc}^{<} \right] \end{aligned} \right] \\ &+ \frac{1}{2} \left[ \begin{aligned} & \frac{1}{i\hbar} \left[ \bar{\Delta}_{hh,cv}^r \vec{S}_{vc}^{<} - \vec{S}_{cv}^{<} \bar{\Delta}_{ee,vc}^a \right] + \frac{1}{i\hbar} \left[ \vec{\delta}_{hh,cv}^r S_{vc,o}^{<} - S_{cv,o}^{<} \vec{\delta}_{ee,vc}^a \right] \\ & + \frac{1}{\hbar} \left[ \vec{\delta}_{hh,cv}^r \times \vec{S}_{vc}^{<} - \vec{S}_{cv}^{<} \times \vec{\delta}_{ee,vc}^a \right] \end{aligned} \right] \\ &+ \frac{1}{2} \left[ \begin{aligned} & + \left[ \bar{\Delta}_{hh,cv}^{<} \vec{S}_{vc}^a - \vec{S}_{cv}^r \bar{\Delta}_{ee,vc}^{<} \right] + \left[ \vec{\delta}_{hh,cv}^{<} S_{vc,o}^a - S_{cv,o}^r \vec{\delta}_{ee,vc}^{<} \right] \\ & + i \left[ \vec{\delta}_{hh,cv}^{<} \times \vec{S}_{vc}^a - \vec{S}_{cv}^r \times \vec{\delta}_{ee,vc}^{<} \right] \end{aligned} \right], \end{aligned} \tag{24}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{cv}^< \\
&= \frac{1}{2} \left[ \left[ \bar{H}_c \vec{S}_{cv}^< - \vec{S}_{cv}^< \bar{H}_v \right] + \left[ \vec{\mathcal{B}}_{cc} S_{cv,o}^< - S_{cv,o}^< \vec{\mathcal{B}}_{vv} \right] \right. \\
&\quad \left. + i \left[ \vec{\mathcal{B}}_{cc} \times \vec{S}_{cv}^< - \vec{S}_{cv}^< \times \vec{\mathcal{B}}_{vv} \right] \right] \\
&\quad \frac{1}{2} \left[ \left[ \bar{\Sigma}_{cc}^r \vec{S}_{cv}^< - \vec{S}_{cc}^< \bar{\Delta}_{hh,cv}^a \right] + \left[ \vec{\Xi}_{cc}^r S_{cv,o}^< - S_{cc,o}^< \vec{\delta}_{hh,cv}^a \right] \right. \\
&\quad \left. + i \left[ \vec{\Xi}_{cc}^r \times \vec{S}_{cv}^< - \vec{S}_{cc}^< \times \vec{\delta}_{hh,cv}^a \right] \right] \\
&\quad \frac{1}{2} \left[ \left[ \bar{\Sigma}_{cc}^< \vec{S}_{cv}^a - \vec{S}_{cc}^r \bar{\Delta}_{hh,cv}^< \right] + \left[ \vec{\Xi}_{cc}^< S_{cv,o}^a - S_{cc,o}^r \vec{\delta}_{hh,cv}^< \right] \right. \\
&\quad \left. + i \left[ \vec{\Xi}_{cc}^< \times \vec{S}_{cv}^a - \vec{S}_{cc}^r \times \vec{\delta}_{hh,cv}^< \right] \right] \\
&\quad \frac{1}{2} \left[ - \left[ \bar{\Delta}_{hh,cv}^r \vec{S}_{vv}^{>T} - \vec{S}_{cv}^< \bar{\Sigma}_{vv}^{rT} \right] - \left[ \vec{\delta}_{hh,cv}^r S_{vv,o}^{>T} - S_{cv,o}^< \vec{\Xi}_{vv}^{rT} \right] \right. \\
&\quad \left. + i \left[ \vec{S}_{cv}^< \times \vec{\Xi}_{vv}^{rT} - \vec{\delta}_{hh,cv}^r \times \vec{S}_{vv}^{>T} \right] \right] \\
&\quad \frac{1}{2} \left[ - \left[ \bar{\Delta}_{hh,cv}^< \vec{S}_{vv}^{rT} - \vec{S}_{cv}^r \bar{\Sigma}_{vv}^{>T} \right] - \left[ \vec{\delta}_{hh,cv}^< S_{vv,o}^{rT} - S_{cv,o}^r \vec{\Xi}_{vv}^{>T} \right] \right. \\
&\quad \left. + i \left[ \vec{S}_{cv}^r \times \vec{\Xi}_{vv}^{>T} - \vec{\delta}_{hh,cv}^< \times \vec{S}_{vv}^{rT} \right] \right], \tag{25}
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{vc}^< \\
&= \frac{1}{2} \left[ \left[ \bar{H}_v \vec{S}_{vc}^< - \vec{S}_{vc}^< \bar{H}_c \right] + \left[ \vec{\mathcal{B}}_{vv} S_{vc,o}^< - S_{vc,o}^< \vec{\mathcal{B}}_{cc} \right] \right. \\
&\quad \left. + i \left[ \vec{\mathcal{B}}_{vv} \times \vec{S}_{vc}^< - \vec{S}_{vc}^< \times \vec{\mathcal{B}}_{cc} \right] \right] \\
&\quad + \frac{1}{2} \left[ - \left[ \bar{\Sigma}_{vv}^{a,T} \vec{S}_{vc}^< - \vec{S}_{vv}^{>T} \bar{\Delta}_{ee,vc}^a \right] - \left[ \vec{\Xi}_{vv}^{a,T} S_{vc,o}^< - S_{vv,o}^{>T} \vec{\delta}_{ee,vc}^a \right] \right. \\
&\quad \left. + i \left[ \vec{S}_{vv}^{>T} \times \vec{\delta}_{ee,vc}^a - \vec{\Xi}_{vv}^{a,T} \times \vec{S}_{vc}^< \right] \right] \\
&\quad + \frac{1}{2} \left[ - \left[ \bar{\Sigma}_{vv}^{>T} \vec{S}_{vc}^a - \vec{S}_{vv}^{aT} \bar{\Delta}_{ee,vc}^< \right] - \left[ \vec{\Xi}_{vv}^{>T} S_{vc,o}^a - S_{vv,o}^{aT} \vec{\delta}_{ee,vc}^< \right] \right. \\
&\quad \left. + i \left[ \vec{S}_{vv}^{aT} \times \vec{\delta}_{ee,vc}^< - \vec{\Xi}_{vv}^{>T} \times \vec{S}_{vc}^a \right] \right] \\
&\quad + \frac{1}{2} \left[ \left[ \bar{\Delta}_{ee,vc}^r \vec{S}_{cc}^< - \vec{S}_{vc}^< \bar{\Sigma}_{cc}^a \right] + \left[ \vec{\delta}_{ee,vc}^r S_{cc,o}^< - S_{vc,o}^< \vec{\Xi}_{cc}^a \right] \right. \\
&\quad \left. + i \left[ \vec{\delta}_{ee,vc}^r \times \vec{S}_{cc}^< - \vec{S}_{vc}^< \times \vec{\Xi}_{cc}^a \right] \right] \\
&\quad + \frac{1}{2} \left[ \left[ \bar{\Delta}_{ee,vc}^< \vec{S}_{cc}^a - \vec{S}_{vc}^r \bar{\Sigma}_{cc}^< \right] + \left[ \vec{\delta}_{ee,vc}^< S_{cc,o}^a - S_{vc,o}^r \vec{\Xi}_{cc}^< \right] \right. \\
&\quad \left. + i \left[ \vec{\delta}_{ee,vc}^< \times \vec{S}_{cc}^a - \vec{S}_{vc}^r \times \vec{\Xi}_{cc}^< \right] \right], \tag{26}
\end{aligned}$$



$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{vv}^< \\
&= -\frac{1}{2} \left\{ \left[ \bar{H}_v, \vec{S}_{vv}^< \right] + \left[ \vec{\mathcal{B}}_{vv}, S_{vv,o}^< \right] + i \left[ \vec{\mathcal{B}}_{vv} \times \vec{S}_{vv}^< - \vec{S}_{vv}^< \times \vec{\mathcal{B}}_{vv} \right] \right\} \\
&- \frac{1}{2} \left[ \left[ \bar{\Sigma}_{vv}^r \vec{S}_{vv}^< - \vec{S}_{vv}^< \bar{\Sigma}_{vv}^a \right] + \left[ \vec{\Xi}_{vv}^r S_{vv,o}^< - S_{vv,o}^< \vec{\Xi}_{vv}^a \right] \right. \\
&\quad \left. + i \left[ \vec{\Xi}_{vv}^r \times \vec{S}_{vv}^< - \vec{S}_{vv}^< \times \vec{\Xi}_{vv}^a \right] \right] \\
&- \frac{1}{2} \left[ \left[ \bar{\Sigma}_{vv}^< \vec{S}_{vv}^a - \vec{S}_{vv}^r \bar{\Sigma}_{vv}^< \right] + \left[ \vec{\Xi}_{vv}^< S_{vv,o}^a - S_{vv,o}^r \vec{\Xi}_{vv}^< \right] \right. \\
&\quad \left. + i \left[ \vec{\Xi}_{vv}^< \times \vec{S}_{vv}^a - \vec{S}_{vv}^r \times \vec{\Xi}_{vv}^< \right] \right] \\
&- \frac{1}{2} \left[ \frac{1}{i\hbar} \left[ \bar{\Delta}_{ee,vc}^r \vec{S}_{cv}^< - \vec{S}_{vc}^< \bar{\Delta}_{hh,cv}^a \right] + \frac{1}{i\hbar} \left[ \vec{\delta}_{ee,vc}^r S_{cv,o}^< - S_{vc,o}^< \vec{\delta}_{hh,cv}^a \right] \right. \\
&\quad \left. + \frac{1}{\hbar} \left[ \vec{\delta}_{ee,vc}^r \times \vec{S}_{cv}^< - \vec{S}_{vc}^< \times \vec{\delta}_{hh,cv}^a \right] \right] \\
&- \frac{1}{2} \left[ \left[ \bar{\Delta}_{ee,vc}^< \vec{S}_{cv}^a - \vec{S}_{vc}^r \bar{\Delta}_{hh,cv}^< \right] + \left[ \vec{\delta}_{ee,vc}^< S_{cv,o}^a - S_{vc,o}^r \vec{\delta}_{hh,cv}^< \right] \right. \\
&\quad \left. + i \left[ \vec{\delta}_{ee,vc}^< \times \vec{S}_{cv}^a - \vec{S}_{vc}^r \times \vec{\delta}_{hh,cv}^< \right] \right]. \tag{27}
\end{aligned}$$

### A. Single Conduction Band Limit

In the single conduction band limit for the electrons, Eq. (24) reduces to

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{cc}^< \\
&= \frac{1}{2} \left[ \bar{H}_c, \vec{S}_{cc}^< \right] + \frac{1}{2} \left[ \vec{\mathcal{B}}_{cc}, S_{cc,o}^< \right] + \frac{i}{2} \left[ \vec{\mathcal{B}}_{cc} \times \vec{S}_{cc}^< - \vec{S}_{cc}^< \times \vec{\mathcal{B}}_{cc} \right] \\
&+ \frac{1}{2} \left[ \left[ \bar{\Sigma}_{cc}^r \vec{S}_{cc}^< - \vec{S}_{cc}^< \bar{\Sigma}_{cc}^a \right] + \left[ \vec{\Xi}_{cc}^r S_{cc,o}^< - S_{cc,o}^< \vec{\Xi}_{cc}^a \right] \right. \\
&\quad \left. + i \left[ \vec{\Xi}_{cc}^r \times \vec{S}_{cc}^< - \vec{S}_{cc}^< \times \vec{\Xi}_{cc}^a \right] \right] \\
&+ \frac{1}{2} \left[ \left[ \bar{\Sigma}_{cc}^< \vec{S}_{cc}^a - \vec{S}_{cc}^r \bar{\Sigma}_{cc}^< \right] + \left[ \vec{\Xi}_{cc}^< S_{cc,o}^a - S_{cc,o}^r \vec{\Xi}_{cc}^< \right] \right. \\
&\quad \left. + i \left[ \vec{\Xi}_{cc}^< \times \vec{S}_{cc}^a - \vec{S}_{cc}^r \times \vec{\Xi}_{cc}^< \right] \right]. \tag{28}
\end{aligned}$$

Upon ignoring the effect of  $S_{cc,o}^{<}$ ,<sup>29</sup> the above further reduces to,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{cc}^{<} \\
&= \frac{1}{2} \left[ \bar{H}_c + \text{Re } \bar{\Sigma}_{cc}^r, \vec{S}_{cc}^{<} \right] + \frac{i}{2} \left\{ \text{Im } \bar{\Sigma}_{cc}^r, \vec{S}_{cc}^{<} \right\} - \frac{i}{2} \left\{ \bar{\Sigma}_{cc}^{<}, \text{Im } \vec{S}_{cc}^r \right\} + \left[ \bar{\Sigma}_{cc}^{<}, \text{Re } \vec{S}_{cc}^r \right] \\
&+ \frac{i}{2} \left[ \vec{\mathcal{B}}_{cc} \times \vec{S}_{cc}^{<} - \vec{S}_{cc}^{<} \times \vec{\mathcal{B}}_{cc} \right] + \frac{i}{2} \left[ \text{Re } \vec{\Xi}_{cc}^r \times \vec{S}_{cc}^{<} - \vec{S}_{cc}^{<} \times \text{Re } \vec{\Xi}_{cc}^r \right] \\
&- \frac{1}{2} \left\{ \text{Im } \vec{\Xi}_{cc}^r \times \vec{S}_{cc}^{<} + \vec{S}_{cc}^{<} \times \text{Im } \vec{\Xi}_{cc}^r \right\} \\
&+ \frac{i}{2} \left[ \vec{\Xi}_{cc}^{<} \times \text{Re } \vec{S}_{cc}^r - \text{Re } \vec{S}_{cc}^r \times \vec{\Xi}_{cc}^{<} \right] + \frac{1}{2} \left\{ \vec{\Xi}_{cc}^{<} \times \text{Im } \vec{S}_{cc}^r + \text{Im } \vec{S}_{cc}^r \times \vec{\Xi}_{cc}^{<} \right\}.
\end{aligned}$$

Further upon neglecting the effects of  $\text{Re } \vec{S}_{cc}^r$ ,<sup>30</sup> as was done in Ref.<sup>11</sup>, we finally obtain

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{cc}^{<} \\
&= \frac{1}{2} \left[ \bar{H}_c + \text{Re } \bar{\Sigma}_{cc}^r, \vec{S}_{cc}^{<} \right] + \frac{i}{2} \left\{ \text{Im } \bar{\Sigma}_{cc}^r, \vec{S}_{cc}^{<} \right\} - \frac{i}{2} \left\{ \bar{\Sigma}_{cc}^{<}, \text{Im } \vec{S}_{cc}^r \right\} \\
&+ \frac{i}{2} \left[ \left( \vec{\mathcal{B}}_{cc} + \text{Re } \vec{\Xi}_{cc}^r \right) \times \vec{S}_{cc}^{<} - \vec{S}_{cc}^{<} \times \left( \vec{\mathcal{B}}_{cc} + \text{Re } \vec{\Xi}_{cc}^r \right) \right] \\
&- \frac{1}{2} \left\{ \text{Im } \vec{\Xi}_{cc}^r \times \vec{S}_{cc}^{<} + \vec{S}_{cc}^{<} \times \text{Im } \vec{\Xi}_{cc}^r \right\} + \frac{1}{2} \left\{ \vec{\Xi}_{cc}^{<} \times \text{Im } \vec{S}_{cc}^r + \text{Im } \vec{S}_{cc}^r \times \vec{\Xi}_{cc}^{<} \right\}.
\end{aligned}$$

Using the following relations,

$$- \text{Im } \vec{\Xi}_{cc}^r = \vec{\gamma}, \quad (29)$$

$$- \text{Im } \bar{\Sigma}_{cc}^r = \bar{\Gamma}, \quad (30)$$

$$- \text{Im } \vec{S}_{cc}^r = \vec{\mathcal{A}}, \quad (31)$$

$$\vec{\mathcal{B}}_{cc} \Rightarrow -\vec{\mathcal{B}}_{cc} = -g_s \frac{e\hbar}{2m^*_c} \vec{B}_{eff} \text{ for electrons.} \quad (32)$$

Then, we obtain

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{cc}^{<} \\
&= \frac{1}{2} \left[ \bar{H}_c + \text{Re } \bar{\Sigma}_{cc}^r, \vec{S}_{cc}^{<} \right] - \frac{i}{2} \left\{ \bar{\Gamma}, \vec{S}_{cc}^{<} \right\} + \frac{i}{2} \left\{ \bar{\Sigma}_{cc}^{<}, \vec{\mathcal{A}} \right\} \\
&+ \frac{i}{2} \left[ \vec{S}_{cc}^{<} \times \left( \vec{\mathcal{B}}_{cc} - \text{Re } \vec{\Xi}_{cc}^r \right) - \left( \vec{\mathcal{B}}_{cc} - \text{Re } \vec{\Xi}_{cc}^r \right) \times \vec{S}_{cc}^{<} \right] \\
&+ \frac{1}{2} \left\{ \vec{\gamma} \times \vec{S}_{cc}^{<} + \vec{S}_{cc}^{<} \times \vec{\gamma} \right\} - \frac{1}{2} \left\{ \vec{\Xi}_{cc}^{<} \times \vec{\mathcal{A}} + \vec{\mathcal{A}} \times \vec{\Xi}_{cc}^{<} \right\}, \quad (33)
\end{aligned}$$

which agrees with the single-band spin-vector magnetization transport equation given in Ref.<sup>11</sup>, within the approximation used and using Eqs. (29)-(32) except for the presence of

the extra term in Eq. (33) given by

$$\frac{i}{2} \left[ \text{Re } \vec{\Xi}_{cc}^r \times \vec{S}_{cc}^< - \vec{S}_{cc}^< \times \text{Re } \vec{\Xi}_{cc}^r \right].$$

This term was overlooked in casting the  $\text{Re } \Sigma^r$  single-particle Hamiltonian  $\tilde{\mathcal{H}}$  in Ref.<sup>11</sup> into its  $2 \times 2$  spin canonical form. Equation (28) is the exact expression for the single-band spin magnetization vector transport equation.

## VI. PSEUDO-SPIN CORRELATION FUNCTIONS, $S_{\alpha\beta,o}^<$

In addition to the twelve multi-band magnetization transport equations of the Pauli-Dirac spin components, we still have four more equations for the  $S_{\alpha\beta,o}^<$  making a total of 16 equations. By the same token as was done for the  $2 \times 2$  real-spin matrices, we can separate the  $2 \times 2$  matrix containing  $S_{\alpha\beta,o}^<$  into a term independent of the band indices, the total charge in the system, plus the pseudo-spin magnetization terms. To do this, we need to revert to the electron picture as discussed before. Then, the additional magnetization transport equations has to do with pseudo-spin magnetization by virtue of our two-band model, i.e., the presence of two-band discrete quantum labels  $v$  and  $c$ .

From Eqs. (20), (21), (22), and (23), we have the following four transport equations for the Pauli-Dirac spin scalar  $S_{\alpha\beta,o}^<$ , where the subscripts  $\alpha$  and  $\beta$  have the range on the set  $\{v, c\}$ ,

$$\begin{aligned} & i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{cc,o}^< \\ &= \frac{1}{2} [\bar{H}_c, S_{cc,o}^<] + \frac{1}{2} [\vec{B}_{cc} \cdot \vec{S}_{cc}^< - \vec{S}_{cc}^< \cdot \vec{B}_{cc}] \\ &+ \frac{1}{2} [\bar{\Sigma}_{cc}^r S_{cc,o}^< - S_{cc,o}^< \bar{\Sigma}_{cc}^a] + \frac{1}{2} [\vec{\Xi}_{cc}^r \cdot \vec{S}_{cc}^< - \vec{S}_{cc}^< \cdot \vec{\Xi}_{cc}^a] \\ &+ \frac{1}{2} [\bar{\Sigma}_{cc}^< S_{cc,o}^a - S_{cc,o}^r \bar{\Sigma}_{cc}^<] + \frac{1}{2} [\vec{\Xi}_{cc}^< \cdot \vec{S}_{cc}^a - \vec{S}_{cc}^r \cdot \vec{\Xi}_{cc}^<] \\ &+ \frac{1}{2} [\bar{\Delta}_{hh,cv}^r S_{vc,o}^< - S_{cv,o}^< \bar{\Delta}_{ee,vc}^a] + \frac{1}{2} [\vec{\delta}_{hh,cv}^r \cdot \vec{S}_{vc}^< - \vec{S}_{cv}^< \cdot \vec{\delta}_{ee,vc}^a] \\ &+ \frac{1}{2} [\bar{\Delta}_{hh,cv}^< S_{vc,o}^a - S_{cv,o}^r \bar{\Delta}_{ee,vc}^<] + \frac{1}{2} [\vec{\delta}_{hh,cv}^< \cdot \vec{S}_{vc}^a - \vec{S}_{cv}^r \cdot \vec{\delta}_{ee,vc}^<], \end{aligned} \quad (34)$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{cv,o}^< \\
&= \frac{1}{2} \left[ \vec{\mathcal{B}}_{cc} \cdot \vec{S}_{cv}^< - \vec{S}_{cv}^< \cdot \vec{\mathcal{B}}_{vv} \right] + \frac{1}{2} \left[ \bar{H}_c S_{cv,o}^< - S_{cv,o}^< \bar{H}_v \right] \\
&+ \frac{1}{2} \left[ \vec{\Xi}_{cc}^r \cdot \vec{S}_{cv}^< - \vec{S}_{cc}^< \cdot \vec{\delta}_{hh,cv}^a \right] + \frac{1}{2} \left[ \bar{\Sigma}_{cc}^r S_{cv,o}^< - S_{cc,o}^< \bar{\Delta}_{hh,cv}^a \right] \\
&+ \frac{1}{2} \left[ \vec{\Xi}_{cc}^< \cdot \vec{S}_{cv}^a - \vec{S}_{cc}^r \cdot \vec{\delta}_{hh,cv}^< \right] + \frac{1}{2} \left[ \bar{\Sigma}_{cc}^< S_{cv,o}^a - S_{cc,o}^r \bar{\Delta}_{hh,cv}^< \right] \\
&- \frac{1}{2} \left[ \bar{\Delta}_{hh,cv}^r S_{vv,o}^{>T} - S_{cv,o}^< \bar{\Sigma}_{vv}^{rT} \right] - \frac{1}{2} \left[ \vec{\delta}_{hh,cv}^r \cdot \vec{S}_{vv}^{>T} - \vec{S}_{cv}^< \cdot \vec{\Xi}_{vv}^{rT} \right] \\
&- \frac{1}{2} \left[ \bar{\Delta}_{hh,cv}^< S_{vv,o}^{rT} - S_{cv,o}^r \bar{\Sigma}_{vv}^{>T} \right] - \frac{1}{2} \left[ \vec{\delta}_{hh,cv,z}^< \cdot \vec{S}_{z,vv}^{rT} - \vec{S}_{z,cv}^r \cdot \vec{\Xi}_{vv,z}^{>T} \right], \tag{35}
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{vc,o}^< \\
&= \frac{1}{2} \left[ \bar{H}_v S_{vc,o}^< - S_{vc,o}^< \bar{H}_c \right] + \frac{1}{2} \left[ \vec{\mathcal{B}}_{vv} \cdot \vec{S}_{vc}^< - \vec{S}_{vc}^< \cdot \vec{\mathcal{B}}_{cc} \right] \\
&- \frac{1}{2} \left[ \bar{\Sigma}_{vv}^{aT} S_{vc,o}^< - S_{vv,o}^{>T} \bar{\Delta}_{ee,vc}^a \right] - \frac{1}{2} \left[ \vec{\Xi}_{vv}^{aT} \cdot \vec{S}_{vc}^< - \vec{S}_{vv}^{>T} \cdot \vec{\delta}_{ee,vc}^a \right] \\
&- \frac{1}{2} \left[ \bar{\Sigma}_{vv}^{>T} S_{vc,o}^a - S_{vv,o}^{aT} \bar{\Delta}_{ee,vc}^< \right] - \frac{1}{2} \left[ \vec{\Xi}_{vv}^{>T} \cdot \vec{S}_{vc}^a - \vec{S}_{vv}^{aT} \cdot \vec{\delta}_{ee,vc}^< \right] \\
&+ \frac{1}{2} \left[ \bar{\Delta}_{ee,vc}^r S_{cc,o}^< - S_{vc,o}^< \bar{\Sigma}_{cc}^a \right] + \frac{1}{2} \left[ \vec{\delta}_{ee,vc}^r \cdot \vec{S}_{cc}^< - \vec{S}_{vc}^< \cdot \vec{\Xi}_{cc}^a \right] \\
&+ \frac{1}{2} \left[ \bar{\Delta}_{ee,vc}^< S_{cc,o}^a - S_{vc,o}^r \bar{\Sigma}_{cc}^< \right] + \frac{1}{2} \left[ \vec{\delta}_{ee,vc,z}^< \cdot \vec{S}_{z,cc}^a - \vec{S}_{z,vc}^r \cdot \vec{\Xi}_{cc,z}^< \right], \tag{36}
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{vv,o}^< \\
&= \frac{1}{2} \left[ \bar{H}_v S_{vv,o}^< - S_{vv,o}^< \bar{H}_v \right] + \frac{1}{2} \left[ \vec{\mathcal{B}}_{vv} \cdot \vec{S}_{vv}^< - \vec{S}_{vv}^< \cdot \vec{\mathcal{B}}_{vv} \right] \\
&+ \frac{1}{2} \left[ \bar{\Sigma}_{vv}^r S_{vv,o}^< - S_{vv,o}^< \bar{\Sigma}_{vv}^a \right] + \frac{1}{2} \left[ \vec{\Xi}_{vv}^r \cdot \vec{S}_{vv}^< - \vec{S}_{vv}^< \cdot \vec{\Xi}_{vv}^a \right] \\
&+ \frac{1}{2} \left[ \bar{\Sigma}_{vv}^< S_{vv,o}^a - S_{vv,o}^r \bar{\Sigma}_{vv}^< \right] + \frac{1}{2} \left[ \vec{\Xi}_{vv}^< \cdot \vec{S}_{vv}^a - \vec{S}_{vv}^r \cdot \vec{\Xi}_{vv}^< \right] \\
&+ \frac{1}{2} \left[ \bar{\Delta}_{ee,vc}^r S_{cv,o}^< - S_{vc,o}^< \bar{\Delta}_{hh,cv}^a \right] + \frac{1}{2} \left[ \vec{\delta}_{ee,vc}^r \cdot \vec{S}_{cv}^< - \vec{S}_{vc}^< \cdot \vec{\delta}_{hh,cv}^a \right] \\
&+ \frac{1}{2} \left[ \bar{\Delta}_{ee,vc}^< S_{cv,o}^a - S_{vc,o}^r \bar{\Delta}_{hh,cv}^< \right] + \frac{1}{2} \left[ \vec{\delta}_{ee,vc}^< \cdot \vec{S}_{cv}^a - \vec{S}_{vc}^r \cdot \vec{\delta}_{hh,cv}^< \right]. \tag{37}
\end{aligned}$$

Note that Eqs. (24)-(27) and Eqs. (34)-(37) yield 16 coupled transport equations. By separating  $S_{\alpha\beta,o}^<$  into the total charge in the system plus the pseudo-spin, the multi-band spin magnetization quantum transport equations nonlinearly incorporates the effects of pseudo-spin magnetization. We have already perform the sign change of Eq. (10) to obtain the

electron picture of Eq. (37) above. What remains to be done for Eqs. (35) and (36) in order to revert to electron picture is to replaced all the transposed quantities,  $F_{vv}^{\alpha T}$  with their equivalent expressions given in the Table 1-3.

### A. Spinless Two-Level Atom and Flat-Band Limits

We observe that in Eqs. (34)-(37) each equation contains spin-independent terms and corresponding terms which involved spin-vector dot products expressing the summation (i.e., the process of 'integrating out') of the real-spin degree of freedom. To gain some understanding into these equations, we examine Eqs. (34)-(37) by first retaining only the spin-independent terms, i.e., for the moment ignoring the vector dot product portions.

We have the resulting transport equations for the scalar  $S_{\alpha\beta,o}^<$  involving only the spin-independent terms of Eqs. (34)-(37),

$$\begin{aligned} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{cc,o}^< &= \frac{1}{2} [\bar{H}_c, S_{cc,o}^<] \\ &+ \frac{1}{2} [\bar{\Delta}_{hh,cv}^r S_{vc,o}^< - S_{cv,o}^< \bar{\Delta}_{ee,vc}^a] + \frac{1}{2} [\bar{\Delta}_{hh,cv}^< S_{vc,o}^a - S_{cv,o}^r \bar{\Delta}_{ee,vc}^<] \\ &+ \frac{1}{2} [\bar{\Sigma}_{cc}^r S_{cc,o}^< - S_{cc,o}^< \bar{\Sigma}_{cc}^a] + \frac{1}{2} [\bar{\Sigma}_{cc}^< S_{cc,o}^a - S_{cc,o}^r \bar{\Sigma}_{cc}^<], \end{aligned} \quad (38)$$

$$\begin{aligned} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{cv,o}^< &= \frac{1}{2} [\bar{H}_c S_{cv,o}^< - S_{cv,o}^< \bar{H}_v] \\ &- \frac{1}{2} [\bar{\Delta}_{hh,cv}^r S_{vv,o}^{>T} + S_{cc,o}^< \bar{\Delta}_{hh,cv}^a] - \frac{1}{2} [\bar{\Delta}_{hh,cv}^< S_{vv,o}^{rT} + S_{cc,o}^r \bar{\Delta}_{hh,cv}^<] \\ &+ \frac{1}{2} [\bar{\Sigma}_{cc}^r S_{cv,o}^< + S_{cv,o}^< \bar{\Sigma}_{vv}^{rT}] + \frac{1}{2} [\bar{\Sigma}_{cc}^< S_{cv,o}^a + S_{cv,o}^r \bar{\Sigma}_{vv}^{>T}], \end{aligned} \quad (39)$$

$$\begin{aligned} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{vc,o}^< &= \frac{1}{2} [\bar{H}_v S_{vc,o}^< - S_{vc,o}^< \bar{H}_c] \\ &+ \frac{1}{2} [\bar{\Delta}_{ee,vc}^r S_{cc,o}^< + S_{vv,o}^{>T} \bar{\Delta}_{ee,vc}^a] + \frac{1}{2} [\bar{\Delta}_{ee,vc}^< S_{cc,o}^a + S_{vv,o}^{aT} \bar{\Delta}_{ee,vc}^<] \\ &- \frac{1}{2} [\bar{\Sigma}_{vv}^{aT} S_{vc,o}^< + S_{vc,o}^< \bar{\Sigma}_{cc}^a] - \frac{1}{2} [\bar{\Sigma}_{vv}^{>T} S_{vc,o}^a + S_{vc,o}^r \bar{\Sigma}_{cc}^<], \end{aligned} \quad (40)$$

$$\begin{aligned} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{vv,o}^< &= \frac{1}{2} [\bar{H}_v S_{vv,o}^< - S_{vv,o}^< \bar{H}_v] \\ &+ \frac{1}{2} [\bar{\Delta}_{ee,vc}^r S_{cv,o}^< - S_{vc,o}^< \bar{\Delta}_{hh,cv}^a] + \frac{1}{2} [\bar{\Delta}_{ee,vc}^< S_{cv,o}^a - S_{vc,o}^r \bar{\Delta}_{hh,cv}^<] \\ &+ \frac{1}{2} [\bar{\Sigma}_{vv}^r S_{vv,o}^< - S_{vv,o}^< \bar{\Sigma}_{vv}^a] + \frac{1}{2} [\bar{\Sigma}_{vv}^< S_{vv,o}^a - S_{vv,o}^r \bar{\Sigma}_{vv}^<]. \end{aligned} \quad (41)$$

To gain insights in these equations, as representing the pseudo-spin part of the spin magnetization transport equations, let us reduce these equations to a two-level flat energy bands or a two-level atomic system. Let us take the zero of energy in the middle of the band or energy gap, and let  $\bar{H}_c = -\bar{H}_v = \frac{\hbar\omega_o}{2}$ . We will also ignore terms arising from the particle self-energies,  $\Sigma$ , as well as terms involving  $\bar{\Delta}^{\lessgtr}$ . We identify the interband matrix elements as  $\bar{\Delta}_{ee,vc}^r = \langle v | \mathcal{H}_I | c \rangle$ , and  $\bar{\Delta}_{hh,cv}^a = \langle c | \mathcal{H}_I | v \rangle$ , so that  $\bar{\Delta}_{hh,vc}^a = \bar{\Delta}_{ee,vc}^r$ , and by virtue of the locality of space-time dependence. The above four equations reduce to the following expressions,

$$i\hbar \frac{\partial}{\partial t} S_{cc,o}^< = \frac{1}{2} [\bar{\Delta}_{hh,cv}^r S_{vc,o}^< - S_{cv,o}^< \bar{\Delta}_{ee,vc}^r], \quad (42)$$

$$i\hbar \frac{\partial}{\partial t} S_{cv,o}^< = \frac{1}{2} [\hbar\omega_o S_{cv,o}^< + \bar{\Delta}_{hh,cv}^r (S_{vv,o}^< - S_{cc,o}^<)], \quad (43)$$

$$i\hbar \frac{\partial}{\partial t} S_{vc,o}^< = \frac{1}{2} [-\hbar\omega_o S_{vc,o}^< + \bar{\Delta}_{ee,vc}^r (S_{cc,o}^< - S_{vv,o}^<)], \quad (44)$$

$$i\hbar \frac{\partial}{\partial t} S_{vv,o}^< = \frac{1}{2} [\bar{\Delta}_{ee,vc}^r S_{cv,o}^< - S_{vc,o}^< \bar{\Delta}_{hh,cv}^r], \quad (45)$$

where we made use of the relation:<sup>31</sup>

$$S_{vv,o}^{>T} = -S_{vv,o}^<, \quad (46)$$

in our two-level atomic or flat-band limiting case.<sup>32</sup>

The pseudo-spin correlation functions are derived by first expressing the  $2 \times 2$  matrix in the band indices into a spin-canonical form as,

$$\begin{aligned} \begin{pmatrix} S_{cc,o}^< & S_{cv,o}^< \\ S_{vc,o}^< & S_{vv,o}^< \end{pmatrix} &= \frac{1}{2} (S_{o,o} I + \vec{S}_o \cdot \vec{\sigma}) \\ &= \frac{1}{2} \begin{pmatrix} S_{o,o} + S_{z,o} & S_{x,o} - iS_{y,o} \\ S_{x,o} + iS_{y,o} & S_{o,o} - S_{z,o} \end{pmatrix}, \end{aligned} \quad (47)$$

where,

$$\begin{aligned} S_{x,o} &= (S_{vc,o}^< + S_{cv,o}^<), \\ iS_{y,o} &= (S_{vc,o}^< - S_{cv,o}^<), \\ S_{z,o} &= (S_{cc,o}^< - S_{vv,o}^<), \\ S_{o,o} &= (S_{cc,o}^< + S_{vv,o}^<), \end{aligned} \quad (48)$$

where we drop the ' $<$ ' superscript in the pseudo-spin correlation functions,  $S_{j,o}$ . We note that  $S_{o,o}$  represent the trace of the original  $4 \times 4$  spin matrix for the two bands. Thus,  $S_{o,o}$  represent the total charge of the system which may vary in space and time, with the caveat that the background positive charge have to be subtracted from the charge represented by the correlation density  $S_{o,o}$  to obtain the net charge.

Similarly, we write the 'energy'-matrix in terms of the Pauli matrix as

$$\begin{aligned} \begin{pmatrix} \bar{H}_c & \bar{\Delta}_{hh,cv}^r \\ \bar{\Delta}_{ee,vc}^r & \bar{H}_v \end{pmatrix} &= \frac{1}{2} \left( \check{H}I + \vec{\mathcal{B}} \cdot \vec{\sigma} \right), \\ &= \frac{1}{2} \begin{pmatrix} \check{H} + \mathcal{B}_z & \mathcal{B}_x - i\mathcal{B}_y \\ \mathcal{B}_x + i\mathcal{B}_y & \check{H} - \mathcal{B}_z \end{pmatrix}, \end{aligned} \quad (49)$$

where we defined  $\vec{\mathcal{B}}$  and  $\check{H}$  as,

$$\begin{aligned} \mathcal{B}_x &= (\bar{\Delta}_{ee,vc}^r + \bar{\Delta}_{hh,cv}^r), \\ i\mathcal{B}_y &= (\bar{\Delta}_{ee,vc}^r - \bar{\Delta}_{hh,cv}^r), \\ \mathcal{B}_z &= (\bar{H}_c - \bar{H}_v) = \hbar\omega_o, \\ \check{H} &= (\bar{H}_c + \bar{H}_v) = 0. \end{aligned} \quad (50)$$

Therefore, we have the transport equations for the pseudo-spin correlation functions,

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} S_{x,o} &= \frac{1}{2} \left[ -\hbar\omega_o S_{vc,o}^< + \bar{\Delta}_{ee,vc}^r (S_{cc,o}^< - S_{vv,o}^<) \right] \\ &\quad + \frac{1}{2} \left[ \hbar\omega_o S_{cv,o}^< + \bar{\Delta}_{hh,cv}^r (S_{vv,o}^< - S_{cc,o}^<) \right] \\ &= \frac{i}{2} (\mathcal{B}_y S_{o,z} - \mathcal{B}_z S_{o,y}), \\ i\hbar \frac{\partial}{\partial t} iS_{y,o} &= \frac{1}{2} \left[ -\hbar\omega_o S_{vc,o}^< + \bar{\Delta}_{ee,vc}^r (S_{cc,o}^< - S_{vv,o}^<) \right] \\ &\quad + \frac{1}{2} \left[ -\hbar\omega_o S_{cv,o}^< - \bar{\Delta}_{hh,cv}^r (S_{vv,o}^< - S_{cc,o}^<) \right] \\ &= \frac{1}{2} [\mathcal{B}_x S_{o,z} - \mathcal{B}_z S_{o,x}], \\ i\hbar \frac{\partial}{\partial t} S_{z,o} &= \frac{1}{2} [\bar{\Delta}_{hh,cv}^r S_{vc,o}^< - S_{cv,o}^< \bar{\Delta}_{ee,vc}^r] \\ &\quad + \frac{1}{2} [-\bar{\Delta}_{ee,vc}^r S_{cv,o}^< + S_{vc,o}^< \bar{\Delta}_{hh,cv}^r] \\ &= \frac{i}{2} (\mathcal{B}_x S_{o,y} - \mathcal{B}_y S_{o,x}). \end{aligned}$$

In vector equation form, the pseudo-spin magnetization equation is,

$$\frac{\partial}{\partial t} \vec{S}_o = \frac{1}{\hbar} \vec{\mathcal{B}} \times \frac{1}{2} \vec{S}_o, \quad (51)$$

The above equation can also be written as

$$\begin{aligned} \frac{\partial}{\partial t} \left( \frac{\hbar}{2} \vec{S}_o \right) &= \frac{mc}{2e\hbar} \vec{\mathcal{B}} \times \frac{e}{mc} \left( \frac{\hbar}{2} \vec{S}_o \right), \\ &= \vec{B} \times \vec{\mu}_B \end{aligned} \quad (52)$$

where

$$\frac{e}{mc} \left( \frac{\hbar}{2} \vec{S}_o \right) = \vec{\mu}_B$$

corresponds to the magnetic moment of the pseudo-spin (pseudo-Bohr magneton), and

$$\frac{mc}{2e\hbar} \vec{\mathcal{B}} = \vec{B}$$

has the units of the magnetic field,  $\vec{B}$ . Thus we have realized a pseudo-spin angular momentum  $\frac{\hbar}{2} \vec{S}_o$ , with effective magnetic field,  $\vec{B}$ , determined by the interband terms in  $\Delta$ 's and energy gap, Eq. (50). Equations (51) and (52) are also known as the Bloch equations. Note that  $\vec{S}_o$  rotates about the  $z$ -axis with frequency  $\frac{\mathcal{B}_z}{\hbar} = \omega_o$ <sup>33</sup> in the counterclockwise sense.

Indeed, the pseudo-spin angular momentum has eigenvalues  $\pm \frac{1}{2} \hbar$  by virtue of the two discrete quantum-energy labels, since from Eq. (48) we have,

$$S_{z,o} = 1 \text{ for } S_{vv,o} = 0 \text{ for the excited state,}$$

$$S_{z,o} = -1 \text{ for } S_{cc,o} = 0 \text{ for the unexcited or ground state.}$$

## B. Spin-Independent Contribution

We now consider the full spin-independent terms of Eqs. (38) - (41). The pseudo-spin correlation functions are given by Eq. (48). We expressed in spin-canonical form the following matrix in band indices,

$$\begin{pmatrix} \bar{H}_c + \text{Re } \bar{\Sigma}_{cc}^r & \text{Re } \bar{\Delta}_{hh,cv}^r \\ \text{Re } \bar{\Delta}_{ee,vc}^r & \bar{H}_v + \text{Re } \bar{\Sigma}_{vv}^r \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \bar{H} + \beta_z & \beta_x - i\beta_y \\ \beta_x + i\beta_y & \bar{H} - \beta_z \end{pmatrix}, \quad (53)$$

$$\begin{pmatrix} \text{Im } \bar{\Sigma}_{cc}^r & \text{Im } \bar{\Delta}_{hh,cv}^r \\ \text{Im } \bar{\Delta}_{ee,vc}^r & \text{Im } \bar{\Sigma}_{vv}^r \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \bar{\zeta} + \zeta_z & \zeta_x - i\zeta_y \\ \zeta_x + i\zeta_y & \bar{\zeta} - \zeta_z \end{pmatrix}, \quad (54)$$



$$\begin{pmatrix} \bar{\Sigma}_{cc}^{\leq} & \bar{\Delta}_{hh,cv}^{\leq} \\ \bar{\Delta}_{ee,vc}^{\leq} & \bar{\Sigma}_{vv}^{\leq} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \bar{\zeta}^{\leq} + \zeta_z^{\leq} & \zeta_x^{\leq} - i\zeta_y^{\leq} \\ \zeta_x^{\leq} + i\zeta_y^{\leq} & \bar{\zeta}^{\leq} - \zeta_z^{\leq} \end{pmatrix}. \quad (55)$$

Note the use of caligraphic  $\mathcal{B}$  in Eq. (49) in contrast to the use of the Greek  $\beta$  in Eq. (53).

The calculation is tedious but the results can be expressed in vector form as

$$\begin{aligned} & i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_o \\ &= \frac{1}{4} [\bar{H}, \vec{S}_o] + \frac{1}{4} [\vec{\beta}, S_{o,o}] + \frac{i}{4} [\vec{\beta} \times \vec{S}_o - \vec{S}_o \times \vec{\beta}] \\ &+ \frac{i}{4} [\{\bar{\zeta}, \vec{S}_o\} + \{\vec{\zeta}, S_{o,o}\}] - \frac{1}{4} \{\vec{\zeta} \times \vec{S}_o + \vec{S}_o \times \vec{\zeta}\} \\ &+ \frac{1}{4} [\bar{\zeta}^{\leq}, \text{Re } S_{o,o}^r] + \frac{1}{4} [\bar{\zeta}^{\leq}, \text{Re } \vec{S}_o^r] \\ &+ \frac{i}{4} [\bar{\zeta}^{\leq} \times \text{Re } \vec{S}_o^r - \text{Re } \vec{S}_o^r \times \bar{\zeta}^{\leq}] \\ &- \frac{i}{4} [\{\bar{\zeta}^{\leq}, \text{Im } \vec{S}_o^r\} + \{\vec{\zeta}^{\leq}, \text{Im } S_{o,o}^r\}] \\ &+ \frac{1}{4} \{\bar{\zeta}^{\leq} \times \text{Im } \vec{S}_o^r + \text{Im } \vec{S}_o^r \times \bar{\zeta}^{\leq}\}, \end{aligned} \quad (56)$$

where we use the following defined relation,

$$\begin{aligned} iS_{y,o}^r &= S_{vc,o}^r - S_{cv,o}^r \\ &= i (\text{Re } S_{y,o}^r + i \text{Im } S_{y,o}^r). \end{aligned}$$

### 1. The Equation for the Scalar $S_{o,o}$

We have the equation for  $S_{o,o}$  representing the total charge is given by,

$$\begin{aligned} & i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{o,o} = \frac{1}{4} [\bar{H}, S_{o,o}] + \frac{1}{4} [\bar{\zeta}^{\leq}, \text{Re } S_{o,o}^r] \\ &+ \frac{i}{4} [\{\bar{\zeta}, S_{o,o}\}] - \frac{i}{4} \{\bar{\zeta}^{\leq}, \text{Im } S_{o,o}^r\} \\ &+ \frac{1}{4} [\vec{\beta}^{\cdot psp}, \vec{S}_o] + \frac{i}{4} \{\vec{\zeta}^{\cdot psp}, \vec{S}_o\} \\ &+ \frac{1}{4} [\bar{\zeta}^{\leq \cdot psp}, \text{Re } \vec{S}_o^r] - \frac{i}{4} \{\bar{\zeta}^{\leq \cdot psp}, \text{Im } \vec{S}_o^r\}, \end{aligned} \quad (57)$$

where the superscript  $\cdot psp$  means taking the dot product with respect to the pseudo-spin vector components. One observes that the first four terms in the the right hand side of Eq. (57) exactly correspond to the terms in the equation of the particle or charge correlation function in system without the spin degree of freedom.

### C. Spin-Dependent Contributions

The spin-dependent terms ignored in Sec. VIA represent the direct coupling of pseudo-spin to real (Pauli-Dirac) spins. We extract from Eqs. (34) - (37) the terms involving the dot products of real-spin vectors for their contributions to the pseudo-spin magnetization transport equations. Since the real-spin degree of freedom is essentially integrated out, what is left are the discrete band indices. First, we expressed all  $2 \times 2$  quantities in the band indices into their pseudo-spin canonical matrix form. We make use of the expressions already given by Eqs. (47), (48), (49), and (50) for the pseudo-spin canonical form of the scalar quantities. The vector quantities must also be expressed into their pseudo-spin canonical matrix form in the band indices producing dyadic tensors.

We have for the real-spin vector dot product portion of the pseudo-spin equations, where the dot product can be written for convenience, using the Einstein summation convention for the Pauli-Dirac spin vector dot product,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{cc,o}^{\leq} \\
&= \frac{1}{2} [\mathcal{B}_{cc,i} S_{cc,i}^{\leq} - S_{cc,i}^{\leq} \mathcal{B}_{cc,i}] \\
&+ \frac{1}{2} [\Xi_{cc,i}^r S_{cc,i}^{\leq} - S_{cc,i}^{\leq} \Xi_{cc,i}^a] + \frac{1}{2} [\Xi_{cc,i}^{\leq} S_{cc,i}^a - S_{cc,i}^r \Xi_{cc,i}^{\leq}] \\
&+ \frac{1}{2} [\delta_{hh,cv,i}^r S_{vc,i}^{\leq} - S_{cv,i}^{\leq} \delta_{ee,vc,i}^a] + \frac{1}{2} [\delta_{hh,cv,i}^{\leq} S_{vc,i}^a - S_{cv,i}^r \delta_{ee,vc,i}^{\leq}] \quad , \quad (58)
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{cv,o}^{\leq} \\
&= \frac{1}{2} [(\mathcal{B}_{cc,i} + \Xi_{cc,i}^r) S_{cv,i}^{\leq} - S_{cv,i}^{\leq} (\mathcal{B}_{vv,i} - \Xi_{vv,i}^{rT})] \\
&- \frac{1}{2} [S_{cc,i}^{\leq} \delta_{hh,cv,i}^a + \delta_{hh,cv,i}^r S_{vv,i}^{>T}] + \frac{1}{2} [\Xi_{cc,i}^{\leq} S_{cv,i}^a + S_{cv,i}^r \Xi_{vv,i}^{>T}] \\
&- \frac{1}{2} [\delta_{hh,cv,i}^{\leq} S_{vv,i}^{rT} + S_{cc,i}^r \delta_{hh,cv,i}^{\leq}] \quad , \quad (59)
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{vc,o}^< \\
&= \frac{1}{2} [(\mathcal{B}_{vv,i} - \Xi_{vv,i}^{aT}) S_{vc,i}^< - S_{vc,i}^< (\mathcal{B}_{cc,i} + \Xi_{cc,i}^a)] \\
&+ \frac{1}{2} [\delta_{ee,vc,i}^r S_{cc,i}^< + S_{vv,i}^{>T} \delta_{ee,vc,i}^a] - \frac{1}{2} [\Xi_{vv,i}^{>T} S_{vc,i}^a + S_{z,vc,i}^r \Xi_{cc,i}^<] \\
&+ \frac{1}{2} [\delta_{ee,vc,i}^< S_{cc,i}^a + S_{vv,i}^{aT} \delta_{ee,vc,i}^<], \tag{60}
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{vv,o}^< \\
&= \frac{1}{2} [\mathcal{B}_{vv,i} S_{vv,i}^< - S_{vv,i}^< \mathcal{B}_{vv,i}] \\
&+ \frac{1}{2} [\Xi_{vv,i}^r S_{vv,i}^< - S_{vv,i}^< \Xi_{vv,i}^a] + \frac{1}{2} [\Xi_{vv,i}^< S_{vv,i}^a - S_{vv,i}^r \Xi_{vv,i}^<] \\
&+ \frac{1}{2} [\delta_{ee,vc,i}^r S_{cv,i}^< - S_{vc,i}^< \delta_{hh,cv,i}^a] + \frac{1}{2} [\delta_{ee,vc,i}^< S_{cv,i}^a - S_{vc,i}^r \delta_{hh,cv,i}^<]. \tag{61}
\end{aligned}$$

We have the following pseudo-spin canonical matrix form for the real-spin vectors labeled by the discrete band indices,

$$\begin{pmatrix} \mathcal{B}_{cc,i} + \text{Re } \Sigma_{cc,i}^r & \text{Re } \delta_{cv,i}^r \\ \text{Re } \delta_{vc,i}^r & \mathcal{B}_{vv,i} + \text{Re } \Sigma_{vv,i}^r \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \bar{\mathcal{B}}_i + \mathcal{B}_{z,i} & \mathcal{B}_{x,i} - i\mathcal{B}_{y,i} \\ \mathcal{B}_{x,i} + i\mathcal{B}_{y,i} & \bar{\mathcal{B}}_i - \mathcal{B}_{z,i} \end{pmatrix}, \tag{62}$$

$$\begin{pmatrix} \text{Im } \Sigma_{cc,i}^r & \text{Im } \delta_{hh,cv,i}^r \\ \text{Im } \delta_{ee,vc,i}^r & \text{Im } \Sigma_{vv,i}^r \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \bar{\zeta}_i + \zeta_{z,i} & \zeta_{x,i} - i\zeta_{y,i} \\ \zeta_{x,i} + i\zeta_{y,i} & \bar{\zeta}_i - \zeta_{z,i} \end{pmatrix}, \tag{63}$$

$$\begin{pmatrix} \Sigma_{cc,i}^< & \delta_{hh,cv,i}^< \\ \delta_{ee,vc,i}^< & \Sigma_{vv,i}^< \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \bar{\zeta}_i^< + \zeta_{z,i}^< & \zeta_{x,i}^< - i\zeta_{y,i}^< \\ \zeta_{x,i}^< + i\zeta_{y,i}^< & \bar{\zeta}_i^< - \zeta_{z,i}^< \end{pmatrix}, \tag{64}$$

$$\begin{aligned}
S_{\alpha\beta,i} &= \begin{pmatrix} S_{cc,i}^< & S_{cv,i}^< \\ S_{vc,i}^< & S_{vv,i}^< \end{pmatrix} \\
&= \frac{1}{2} \begin{pmatrix} \bar{S}_i + S_{z,i} & S_{x,i} - iS_{y,i} \\ S_{x,i} + iS_{y,i} & \bar{S}_i - S_{z,i} \end{pmatrix}. \tag{65}
\end{aligned}$$

We will also make use of the definition

$$\begin{aligned}
iS_{y,i}^r &= i(\text{Re } S_{y,i}^r + i \text{Im } S_{y,i}^r) \\
&= S_{vc,i}^r - S_{cv,i}^r \tag{66}
\end{aligned}$$

The results for the pseudo-spin magnetization equation from the contributions of real-spin vector dot product terms in Eqs. (58) - (61) can also be written as a pseudo-spin vector equation similar to the scalar-contribution given by Eq. (56) as,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_o \\
&= \frac{1}{4} [\vec{\mathcal{B}}_i, \vec{S}_i] + \frac{1}{4} [\vec{\mathcal{B}}_i, \vec{S}_i] + \frac{i}{4} [\vec{\mathcal{B}}_i \times \vec{S}_i - \vec{S}_i \times \vec{\mathcal{B}}_i] \\
&+ \frac{i}{4} [\{\bar{\zeta}_i, \vec{S}_i\} + \{\vec{\zeta}_i, \bar{S}_i\}] - \frac{1}{4} \{\vec{\zeta}_i \times \vec{S}_i + \vec{S}_i \times \vec{\zeta}_i\} \\
&+ \frac{1}{4} [\vec{\zeta}_i^<, \text{Re } \bar{S}_i^r] + \frac{1}{4} [\bar{\zeta}_i^<, \text{Re } \vec{S}_i^r] \\
&+ \frac{i}{4} [\vec{\zeta}_i^< \times \text{Re } \bar{S}_i^r - \text{Re } \vec{S}_i^r \times \bar{\zeta}_i^<] \\
&- \frac{i}{4} [\{\bar{\zeta}_i^<, \text{Im } \bar{S}_i^r\} + \{\vec{\zeta}_i^<, \text{Im } \vec{S}_i^r\}] \\
&+ \frac{1}{4} \{\vec{\zeta}_i^< \times \text{Im } \bar{S}_i^r + \text{Im } \vec{S}_i^r \times \bar{\zeta}_i^<\}.
\end{aligned} \tag{67}$$

### 1. The Equation for the Scalar $S_{o,o}$

Similarly the contribution to the equation for  $S_{o,o}$  goes as,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{o,o} \\
&= \frac{1}{4} [\vec{\mathcal{B}}_i, \bar{S}_i] + \frac{1}{4} [\vec{\mathcal{B}}_i^{psp}, \vec{S}_i] \\
&+ \frac{1}{4} i [\{\bar{\zeta}_i, \bar{S}_i\}] + \frac{1}{4} i \{\vec{\zeta}_i^{psp}, \vec{S}_i\} \\
&+ \frac{1}{4} [\bar{\zeta}_i^<, \text{Re } \bar{S}_i^r] - \frac{1}{4} i \{\bar{\zeta}_i^<, \text{Im } \bar{S}_i^r\} \\
&+ \frac{1}{4} [\vec{\zeta}_i^{<psp}, \text{Re } \vec{S}_i^r] - \frac{1}{4} i \{\vec{\zeta}_i^{<psp}, \text{Im } \vec{S}_i^r\},
\end{aligned} \tag{68}$$

where the superscripts  $^{psp}$  indicates the process of taking the pseudo-spin vector dot products or the process of 'integrating out' the pseudo-spin degree of freedoms, aside from the dot product of the Pauli-Dirac spin vector indicated by the Einstein summation convention for the repeated ' $i$ ' index.

## VII. NONEQUILIBRIUM PSEUDO-SPIN EQUATIONS

The nonequilibrium pseudo-spin vector transport equations result by adding Eqs. (56) and (67)

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_o \\
&= \frac{1}{4} [\bar{H}, \vec{S}_o] + \frac{1}{4} [\bar{\zeta}^<, \text{Re } \vec{S}_o^r] + \frac{i}{4} \{ \bar{\zeta}, \vec{S}_o \} - \frac{i}{4} \{ \bar{\zeta}^<, \text{Im } \vec{S}_o^r \} \\
&+ \frac{1}{2} [\vec{\beta}, S_{o,o}] + \frac{1}{4} [\vec{\zeta}^<, \text{Re } S_{o,o}^r] + \frac{i}{4} \{ \vec{\zeta}, S_{o,o} \} - \frac{i}{4} \{ \vec{\zeta}^<, \text{Im } S_{o,o}^r \} \\
&+ \frac{i}{4} [\vec{\beta} \times \vec{S}_o - \vec{S}_o \times \vec{\beta}] + \frac{i}{4} [\vec{\zeta}^< \times \text{Re } \vec{S}_o^r - \text{Re } \vec{S}_o^r \times \vec{\zeta}^<] \\
&- \frac{1}{4} \{ \vec{\zeta} \times \vec{S}_o + \vec{S}_o \times \vec{\zeta} \} + \frac{1}{4} \{ \vec{\zeta}^< \times \text{Im } \vec{S}_o^r + \text{Im } \vec{S}_o^r \times \vec{\zeta}^< \} \\
&+ \frac{1}{4} [\bar{\mathcal{B}}_i, \vec{S}_i] + \frac{1}{4} [\bar{\zeta}_i^<, \text{Re } \vec{S}_i^r] + \frac{i}{4} \{ \bar{\zeta}_i, \vec{S}_i \} - \frac{i}{4} \{ \bar{\zeta}_i^<, \text{Im } \vec{S}_i^r \} \\
&+ \frac{1}{4} [\vec{\mathcal{B}}_i, \bar{S}_i] + \frac{1}{4} [\vec{\zeta}_i^<, \text{Re } \bar{S}_i^r] + \frac{i}{4} \{ \vec{\zeta}_i, \bar{S}_i \} - \frac{i}{4} \{ \vec{\zeta}_i^<, \text{Im } \bar{S}_i^r \} \\
&+ \frac{i}{4} [\vec{\mathcal{B}}_i \times \vec{S}_i - \vec{S}_i \times \vec{\mathcal{B}}_i] + \frac{i}{4} [\vec{\zeta}_i^< \times \text{Re } \vec{S}_i^r - \text{Re } \vec{S}_i^r \times \vec{\zeta}_i^<] \\
&- \frac{1}{4} \{ \vec{\zeta}_i \times \vec{S}_i + \vec{S}_i \times \vec{\zeta}_i \} + \frac{1}{4} \{ \vec{\zeta}_i^< \times \text{Im } \vec{S}_i^r + \text{Im } \vec{S}_i^r \times \vec{\zeta}_i^< \}.
\end{aligned} \tag{69}$$

The corresponding absolute scalar<sup>34</sup> equation is obtained by adding Eqs. (57) and (68),

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) S_{o,o} \\
&= \frac{1}{4} [\bar{H}, S_{o,o}] + \frac{1}{4} [\bar{\zeta}^<, \text{Re } S_{o,o}^r] + \frac{i}{4} [\{ \bar{\zeta}, S_{o,o} \}] - \frac{i}{4} \{ \bar{\zeta}^<, \text{Im } S_{o,o}^r \} \\
&+ \frac{1}{4} [\vec{\beta}^{\text{psp}}, \vec{S}_o] + \frac{1}{4} [\vec{\zeta}^< \cdot \text{psp}, \text{Re } \vec{S}_o^r] + \frac{i}{4} \{ \vec{\zeta}^{\text{psp}}, \vec{S}_o \} - \frac{i}{4} \{ \vec{\zeta}^< \cdot \text{psp}, \text{Im } \vec{S}_o^r \} \\
&+ \frac{1}{4} [\bar{\mathcal{B}}_i, \bar{S}_i] + \frac{1}{4} [\bar{\zeta}_i^<, \text{Re } \bar{S}_i^r] + \frac{i}{4} \{ \bar{\zeta}_i, \bar{S}_i \} - \frac{i}{4} \{ \bar{\zeta}_i^<, \text{Im } \bar{S}_i^r \} \\
&+ \frac{1}{4} [\vec{\mathcal{B}}_i^{\text{psp}}, \vec{S}_i] + \frac{1}{4} [\vec{\zeta}_i^< \cdot \text{psp}, \text{Re } \vec{S}_i^r] + \frac{i}{4} \{ \vec{\zeta}_i^{\text{psp}}, \vec{S}_i \} - \frac{i}{4} \{ \vec{\zeta}_i^< \cdot \text{psp}, \text{Im } \vec{S}_i^r \}.
\end{aligned} \tag{70}$$

Equations (24) - (27) and (69) - (70) constitute the nonequilibrium multi-band magnetization quantum transport equations. These sixteen equations for the components of the spin vectors are the main results of this paper.

## VIII. MULTI-BAND SPIN QUANTUM TRANSPORT EQUATIONS

We summarize the main results off this paper by recasting the pertinent equations into more meaningful expressions. We start by rewriting the RHS of Pauli-Dirac SMQTEs into three groups as,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{cc}^< \\
&= +\frac{1}{2} \left[ \begin{aligned} & \left[ \bar{H}_c, \vec{S}_{cc}^< \right] \\ & + \left[ (\bar{\Sigma}_{cc}^r) \vec{S}_{cc}^< - \vec{S}_{cc}^< (\bar{\Sigma}_{cc}^a) \right] + \left[ \bar{\Sigma}_{cc}^< (\vec{S}_{cc}^a) - (\vec{S}_{cc}^r) \bar{\Sigma}_{cc}^< \right] \\ & + \left[ (\bar{\Delta}_{hh,cv}^r) \vec{S}_{vc}^< - \vec{S}_{cv}^< (\bar{\Delta}_{ee,vc}^a) \right] + \left[ \bar{\Delta}_{hh,cv}^< \vec{S}_{vc}^a - \vec{S}_{cv}^r \bar{\Delta}_{ee,vc}^< \right] \end{aligned} \right] \\
&+ \frac{1}{2} \left[ \begin{aligned} & +i \left[ \vec{\mathcal{B}}_{cc} \times \vec{S}_{cc}^< - \vec{S}_{cc}^< \times \vec{\mathcal{B}}_{cc} \right] \\ & +i \left[ \left( \vec{\Xi}_{cc}^r \right) \times \vec{S}_{cc}^< - \vec{S}_{cc}^< \times \left( \vec{\Xi}_{cc}^a \right) \right] + i \left[ \vec{\Xi}_{cc}^< \times \left( \vec{S}_{cc}^a \right) - \left( \vec{S}_{cc}^r \right) \times \vec{\Xi}_{cc}^< \right] \\ & +i \left[ \vec{\delta}_{hh,cv}^r \times \vec{S}_{vc}^< - \vec{S}_{cv}^< \times \vec{\delta}_{ee,vc}^a \right] + i \left[ \vec{\delta}_{hh,cv}^< \times \vec{S}_{vc}^a - \vec{S}_{cv}^r \times \vec{\delta}_{ee,vc}^< \right] \end{aligned} \right] \\
&+ \frac{1}{2} \left[ \begin{aligned} & + \left[ \vec{\mathcal{B}}_{cc}, S_{cc,o}^< \right] \\ & + \left[ \left( \vec{\Xi}_{cc}^r \right) S_{cc,o}^< - S_{cc,o}^< \left( \vec{\Xi}_{cc}^a \right) \right] + \left[ \vec{\Xi}_{cc}^< (S_{cc,o}^a) - (S_{cc,o}^r) \vec{\Xi}_{cc}^< \right] \\ & + \left[ \left( \vec{\delta}_{hh,cv}^r \right) S_{vc,o}^< - S_{cv,o}^< \left( \vec{\delta}_{ee,vc}^a \right) \right] + \left[ \vec{\delta}_{hh,cv}^< S_{vc,o}^a - S_{cv,o}^r \vec{\delta}_{ee,vc}^< \right] \end{aligned} \right], \tag{71}
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{cv}^< \\
&= \frac{1}{2} \left[ \begin{aligned} & \left[ \bar{H}_c \vec{S}_{cv}^< - \vec{S}_{cv}^< \bar{H}_v \right] \\ & + \left[ \bar{\Sigma}_{cc}^r \vec{S}_{cv}^< - \vec{S}_{cc}^< \bar{\Delta}_{hh,cv}^a \right] + \left[ \bar{\Sigma}_{cc}^< \vec{S}_{cv}^a - \vec{S}_{cc}^r \bar{\Delta}_{hh,cv}^< \right] \\ & - \left[ \bar{\Delta}_{hh,cv}^r \vec{S}_{vv}^{>T} - \vec{S}_{cv}^< \bar{\Sigma}_{vv}^{rT} \right] - \left[ \bar{\Delta}_{hh,cv}^< \vec{S}_{vv}^{rT} - \vec{S}_{cv}^r \bar{\Sigma}_{vv}^{>T} \right] \end{aligned} \right] \\
&+ \frac{1}{2} \left[ \begin{aligned} & +i \left[ \vec{\mathcal{B}}_{cc} \times \vec{S}_{cv}^< - \vec{S}_{cv}^< \times \vec{\mathcal{B}}_{vv} \right] \\ & +i \left[ \vec{\Xi}_{cc}^r \times \vec{S}_{cv}^< - \vec{S}_{cc}^< \times \vec{\delta}_{hh,cv}^a \right] + i \left[ \vec{\Xi}_{cc}^< \times \vec{S}_{cv}^a - \vec{S}_{cc}^r \times \vec{\delta}_{hh,cv}^< \right] \\ & -i \left[ \vec{\delta}_{hh,cv}^r \times \vec{S}_{vv}^{>T} - \vec{S}_{cv}^< \times \vec{\Xi}_{vv}^{rT} \right] - i \left[ \vec{\delta}_{hh,cv}^< \times \vec{S}_{vv}^{rT} - \vec{S}_{cv}^r \times \vec{\Xi}_{vv}^{>T} \right] \end{aligned} \right] \\
&+ \frac{1}{2} \left[ \begin{aligned} & + \left[ \vec{\mathcal{B}}_{cc} S_{cv,o}^< - S_{cv,o}^< \vec{\mathcal{B}}_{vv} \right] \\ & + \left[ \vec{\Xi}_{cc}^r S_{cv,o}^< - S_{cc,o}^< \vec{\delta}_{hh,cv}^a \right] + \left[ \vec{\Xi}_{cc}^< S_{cv,o}^a - S_{cc,o}^r \vec{\delta}_{hh,cv}^< \right] \\ & - \left[ \vec{\delta}_{hh,cv}^r S_{vv,o}^{>T} - S_{cv,o}^< \vec{\Xi}_{vv}^{rT} \right] - \left[ \vec{\delta}_{hh,cv}^< S_{vv,o}^{rT} - S_{cv,o}^r \vec{\Xi}_{vv}^{>T} \right] \end{aligned} \right], \tag{72}
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{vc}^< \\
&= \frac{1}{2} \left[ \begin{aligned} & \left[ \bar{H}_v \vec{S}_{vc}^< - \vec{S}_{vc}^< \bar{H}_c \right] \\ & - \left[ \bar{\Sigma}_{vv}^{a,T} \vec{S}_{vc}^< - \vec{S}_{vv}^{>T} \bar{\Delta}_{ee,vc}^a \right] - \left[ \bar{\Sigma}_{vv}^{>T} \vec{S}_{vc}^a - \vec{S}_{vv}^{aT} \bar{\Delta}_{ee,vc}^< \right] \\ & + \left[ \bar{\Delta}_{ee,vc}^r \vec{S}_{cc}^< - \vec{S}_{vc}^< \bar{\Sigma}_{cc}^a \right] + \left[ \bar{\Delta}_{ee,vc}^< \vec{S}_{cc}^a - \vec{S}_{vc}^r \bar{\Sigma}_{cc}^< \right] \end{aligned} \right] \\
&+ \frac{1}{2} \left[ \begin{aligned} & + i \left[ \vec{\mathcal{B}}_{vv} \times \vec{S}_{vc}^< - \vec{S}_{vc}^< \times \vec{\mathcal{B}}_{cc} \right] \\ & + i \left[ \vec{S}_{vv}^{aT} \times \vec{\delta}_{ee,vc}^< - \vec{\Xi}_{vv}^{>T} \times \vec{S}_{vc}^a \right] + i \left[ \vec{S}_{vv}^{>T} \times \vec{\delta}_{ee,vc}^a - \vec{\Xi}_{vv}^{a,T} \times \vec{S}_{vc}^< \right] \\ & + i \left[ \vec{\delta}_{ee,vc}^r \times \vec{S}_{cc}^< - \vec{S}_{vc}^< \times \vec{\Xi}_{cc}^a \right] + i \left[ \vec{\delta}_{ee,vc}^< \times \vec{S}_{cc}^a - \vec{S}_{vc}^r \times \vec{\Xi}_{cc}^< \right] \end{aligned} \right] \\
&+ \frac{1}{2} \left[ \begin{aligned} & + \left[ \vec{\mathcal{B}}_{vv} S_{vc,o}^< - S_{vc,o}^< \vec{\mathcal{B}}_{cc} \right] \\ & - \left[ \vec{\Xi}_{vv}^{a,T} S_{vc,o}^< - S_{vv,o}^{>T} \vec{\delta}_{ee,vc}^a \right] - \left[ \vec{\Xi}_{vv}^{>T} S_{vc,o}^a - S_{vv,o}^{aT} \vec{\delta}_{ee,vc}^< \right] \\ & + \left[ \vec{\delta}_{ee,vc}^r S_{cc,o}^< - S_{vc,o}^< \vec{\Xi}_{cc}^a \right] + \left[ \vec{\delta}_{ee,vc}^< S_{cc,o}^a - S_{vc,o}^r \vec{\Xi}_{cc}^< \right] \end{aligned} \right], \tag{73}
\end{aligned}$$

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_{vv}^< \\
&= \frac{1}{2} \left\{ \begin{aligned} & \left[ \bar{H}_v, \vec{S}_{vv}^< \right] \\ & + \left[ \bar{\Sigma}_{vv}^r \vec{S}_{vv}^< - \vec{S}_{vv}^< \bar{\Sigma}_{vv}^a \right] + \left[ \bar{\Sigma}_{vv}^< \vec{S}_{vv}^a - \vec{S}_{vv}^r \bar{\Sigma}_{vv}^< \right] \\ & + \left[ \bar{\Delta}_{ee,vc}^r \vec{S}_{cv}^< - \vec{S}_{vc}^< \bar{\Delta}_{hh,cv}^a \right] + \left[ \bar{\Delta}_{ee,vc}^< \vec{S}_{cv}^a - \vec{S}_{vc}^r \bar{\Delta}_{hh,cv}^< \right] \end{aligned} \right\} \\
&+ \frac{1}{2} \left[ \begin{aligned} & + i \left[ \vec{\mathcal{B}}_{vv} \times \vec{S}_{vv}^< - \vec{S}_{vv}^< \times \vec{\mathcal{B}}_{vv} \right] \\ & + i \left[ \vec{\Xi}_{vv}^r \times \vec{S}_{vv}^< - \vec{S}_{vv}^< \times \vec{\Xi}_{vv}^a \right] + i \left[ \vec{\Xi}_{vv}^< \times \vec{S}_{vv}^a - \vec{S}_{vv}^r \times \vec{\Xi}_{vv}^< \right] \\ & + i \left[ \vec{\delta}_{ee,vc}^< \times \vec{S}_{cv}^a - \vec{S}_{vc}^r \times \vec{\delta}_{hh,cv}^< \right] + i \left[ \vec{\delta}_{ee,vc}^r \times \vec{S}_{cv}^< - \vec{S}_{vc}^< \times \vec{\delta}_{hh,cv}^a \right] \end{aligned} \right] \\
&+ \frac{1}{2} \left[ \begin{aligned} & + \left[ \vec{\mathcal{B}}_{vv}, S_{vv,o}^< \right] \\ & + \left[ \vec{\Xi}_{vv}^r S_{vv,o}^< - S_{vv,o}^< \vec{\Xi}_{vv}^a \right] + \left[ \vec{\Xi}_{vv}^< S_{vv,o}^a - S_{vv,o}^r \vec{\Xi}_{vv}^< \right] \\ & + \left[ \vec{\delta}_{ee,vc}^< S_{cv,o}^a - S_{vc,o}^r \vec{\delta}_{hh,cv}^< \right] + \left[ \vec{\delta}_{ee,vc}^r S_{cv,o}^< - S_{vc,o}^< \vec{\delta}_{hh,cv}^a \right] \end{aligned} \right]. \tag{74}
\end{aligned}$$

We also write the RHS of the pseudo-spin transport equation into a similar three group

of terms not involving the real spin, plus term involving real spin as,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \vec{S}_o \\
&= \left[ \begin{aligned} & \frac{1}{4} [\vec{H}, \vec{S}_o] + \frac{1}{4} [\vec{\zeta}^<, \text{Re } \vec{S}_o^r] \\ & + \frac{i}{4} \{ \vec{\zeta}, \vec{S}_o \} - \frac{i}{4} \{ \vec{\zeta}^<, \text{Im } \vec{S}_o^r \} \end{aligned} \right] \\
&+ \left[ \begin{aligned} & \frac{i}{4} [\vec{\beta} \times \vec{S}_o - \vec{S}_o \times \vec{\beta}] + \frac{i}{4} [\vec{\zeta}^< \times \text{Re } \vec{S}_o^r - \text{Re } \vec{S}_o^r \times \vec{\zeta}^<] \\ & - \frac{1}{4} \{ \vec{\zeta} \times \vec{S}_o + \vec{S}_o \times \vec{\zeta} \} + \frac{1}{4} \{ \vec{\zeta}^< \times \text{Im } \vec{S}_o^r + \text{Im } \vec{S}_o^r \times \vec{\zeta}^< \} \end{aligned} \right] \\
&+ \left[ \begin{aligned} & \frac{i}{4} [\vec{\mathcal{B}}_i \times \vec{S}_{o,i} - \vec{S}_{o,i} \times \vec{\mathcal{B}}_i] \\ & + \frac{i}{4} [\vec{\zeta}_i^< \times \text{Re } \vec{S}_{o,i}^r - \text{Re } \vec{S}_{o,i}^r \times \vec{\zeta}_i^<] \\ & - \frac{1}{4} \{ \vec{\zeta}_i \times \vec{S}_{o,i} + \vec{S}_{o,i} \times \vec{\zeta}_i \} \\ & + \frac{1}{4} \{ \vec{\zeta}_i^< \times \text{Im } \vec{S}_{o,i}^r + \text{Im } \vec{S}_{o,i}^r \times \vec{\zeta}_i^< \} \end{aligned} \right] \\
&+ \left[ \begin{aligned} & \frac{1}{4} [\vec{\mathcal{B}}_i, \vec{S}_{o,i}] + \frac{1}{4} [\vec{\mathcal{B}}_i, \vec{S}_{o,i}] \\ & + \frac{i}{4} \{ \vec{\zeta}_i, \vec{S}_{o,i} \} + \frac{i}{4} \{ \vec{\zeta}_i, \vec{S}_{o,i} \} \\ & + \frac{1}{4} [\vec{\zeta}_i^<, \text{Re } \vec{S}_{o,i}^r] + \frac{1}{4} [\vec{\zeta}_i^<, \text{Re } \vec{S}_{o,i}^r] \\ & - \frac{i}{4} \{ \vec{\zeta}_i^<, \text{Im } \vec{S}_{o,i}^r \} - \frac{i}{4} \{ \vec{\zeta}_i^<, \text{Im } \vec{S}_{o,i}^r \} \end{aligned} \right] \\
&+ \left[ \begin{aligned} & + \frac{1}{4} [\vec{\beta}, S_{o,o}] + \frac{1}{4} [\vec{\zeta}^<, \text{Re } S_{o,o}^r] \\ & + \frac{i}{4} \{ \vec{\zeta}, S_{o,o} \} - \frac{i}{4} \{ \vec{\zeta}^<, \text{Im } S_{o,o}^r \} \end{aligned} \right]. \tag{75}
\end{aligned}$$

Equations (71) -(75) and (70) constitute the nonequilibrium spin magnetization quantum transport equations of this paper, written into at least three groups of more meaningful terms. One note that the first group of terms of the magnetization vector equations is similar to that of the nonequilibrium spinless particle-correlation transport equations by virtue of the spin-independent transport coefficients, second group contains all the torque terms, and the last group generally contains coupling between spin, pseudo-spin, and charge. Note that the equation for the pseudo-spin magnetization vector involves coupling to the total charge represented by  $S_{o,o}$ .

## IX. RETARDED GREEN'S FUNCTION AND SELF-CONSISTENCY

Clearly, some important observables, described by the spectral correlation function,  $\text{Im } G^r$ , and occupation number correlation function,  $i\hbar G^<$ , are the governing dynamical



variables in quantum transport physics. Thus, strictly speaking one must also solve for the transport equation of the nonequilibrium retarded Green's function and self-energy in the presence of spin. The nonequilibrium quantum superfield formalism yields the equation for  $G^r(1, 2)$  as<sup>4</sup>

$$i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) G^r(1, 2) = [(\bar{v} + \Sigma^r), G^r]_{1,2}, \quad (76)$$

where  $\bar{v}$  is the single-particle Hamiltonian, and all quantum labels are absorbed in the two-point arguments denoted by the numeral 1, 2. Then similar procedure can be followed in bringing the above equation into a multiband matrix form. Then the submatrices can be expressed in the spin-canonical form as in Eq. (15), and one solves for the retarded spin correlation functions. This process will yield another 16 nonequilibrium transport equations.

However, usually one is focused mainly in solving the spin magnetization transport equations, and treat Eq. (76) as separate calculations.<sup>35</sup> In most cases, the solution to Eq. (76) are obtained by some sort of approximation and the results plugged into the spin magnetization equations as was done in the simulation of quantum transport equations for charge carriers in nanoelectronic devices. This is coupled with some further simplification by ignoring the effect of  $\text{Re } S^r$  in the spin magnetization transport equations.<sup>36</sup> For these reasons, we will no longer spend time in rigorously including Eq. (76) in our spin magnetization transport equations.

### A. Self-Consistent Electric Field and Potential

There is also an important ingredient in the nonlinearity of the SMQTEs introduced through the need for the self-consistency of the potential distribution.<sup>37-39</sup> This in turn affects the spin-orbit coupling terms in the single particle Hamiltonian and even affects the corresponding many-body aspects of spin-orbit coupling.<sup>40</sup> Thus one also need to solve the Poisson equation for the self-consistent potential, similar to what was done in the numerical simulation of resonant tunneling devices.<sup>8,9,37</sup> Clearly then the total charge, represented by  $S_{o,o}$  obeying Eq. (70) is also coupled, after correcting for the background positive charge, to the potential through the Poisson equation affecting the single particle Hamiltonian and many-body effects in a highly nonlinear self-consistent loop to be accounted for in real device numerical simulations, where clever approximations are eventually needed to bring the problem to a manageable proportion.

## X. PHASE-SPACE SPIN QUANTUM TRANSPORT EQUATIONS

We will first give formal prescriptions on how to transform all the *nonlocal two space-time-points* correlation-function quantum transport equations, given in Eqs. (24) - (27) and (69) - (70) into kinetic QDF transport equations defined in *local phase-space points*, i.e., in  $(p, q; E, t)$  phase space. This is achieved through the use of Buot's discrete phase-space transformation for condensed matter,<sup>41</sup> or his discrete formulation of quantum mechanics leading to the lattice version of the Weyl transformation. Then we will examine what has so far been attempted in the literature towards the treatment of kinetic QDF transport in spintronic devices.

We observe that all of the terms entering in Eqs. (24) - (27) and (69) - (70) involve commutators and anticommutators of two scalar functions and of a scalar and a vector functions, as well as sum and difference of cross products of vector functions. It is more convenient to cast everything in terms of commutators and anticommutators before making the necessary transformation to kinetic equations in phase space. Thus, we need to change the expressions involving cross product of vectors to commutator or anti-commutator, as the case maybe. We make use of the following identities for two vectors  $\vec{A}$  and  $\vec{B}$ ,

$$\begin{aligned}\vec{A} \times \vec{B} - \vec{B} \times \vec{A} &= \hat{I}_i \hat{\epsilon}_{ijk} \{A_j, B_k\}, \\ \vec{A} \times \vec{B} + \vec{B} \times \vec{A} &= \hat{I}_i \hat{\epsilon}_{ijk} [A_j, B_k],\end{aligned}$$

where  $\hat{I}_i$  is the unit dyadic symmetric tensor or *idemfactor*, and  $\hat{\epsilon}_{ijk}$  is the anti-symmetric unit tensor. As used before, the square bracket stands for the commutator and the curly bracket stands for anti-commutator of the two vector components separated by a comma. We also have terms that goes like,  $\vec{A} \times \vec{B} - \vec{C} \times \vec{D}$  occurring in Eqs. (25) and (26), for example the term  $\vec{\Xi}_{cc}^< \times \vec{S}_{cv}^a - \vec{S}_{cc}^r \times \vec{\delta}_{hh,cv}^<$  in Eqs. (25). For the typical cross products involving four spin vectors in Eq. (77), we will simply expand this as

$$\vec{A} \times \vec{B} - \vec{C} \times \vec{D} = \hat{I}_i \hat{\epsilon}_{ijk} A_j B_k - \hat{I}_i \hat{\epsilon}_{ijk} C_j D_k \quad (77)$$

### A. Transformation to $(p, q, E, t)$ Phase-Space

The QDF kinetic transport equations in  $(p, q, E, t)$ -space are obtained by applying the "lattice" Weyl transformation of the correlation function equations by using the following

set of identities<sup>7</sup> (although continuum approximation is interchangeably used, this is not essential and we adapt the word "lattice" when referring to solid-state problems). For convenience, we give these here the LWtransformation of the following two space-time-point correlation functions:

$$i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) F(12) \Leftrightarrow i\hbar \frac{\partial}{\partial t} F_w(\vec{p}, \vec{q}, E, t), \quad (78)$$

where the Bloch states has the representation,  $\langle q | p \rangle = \left( \frac{1}{N\hbar^3} \right)^{\frac{1}{2}} \exp \frac{i}{\hbar} \{ \vec{p} \cdot \vec{q} - Et \}$  [  $N\hbar^3 \Rightarrow h^3$  in the continuum limit], representing a traveling wave in lattice space, with group velocity in a band  $\lambda$  given by  $\frac{dE_\lambda\{\vec{p}\}}{d\vec{p}}$ . The above identities readily follow from the definition of the LWtransform, which in the continuum approximation can simply be written as

$$\begin{aligned} i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) F(12) &= i\hbar \frac{\partial}{\partial t} \int dv \, d\tau e^{\frac{i}{\hbar}(p \cdot v - E\tau)} F\left(q - \frac{v}{2}, t - \frac{\tau}{2}; q + \frac{v}{2}, t + \frac{\tau}{2}\right) \\ &\Leftrightarrow i\hbar \frac{\partial}{\partial t} F_w(\vec{p}, \vec{q}, E, t). \end{aligned}$$

The second set is the LWtransform of a product of two-point correlation functions [  $p \equiv (\vec{p}, -E)$  and  $q = (\vec{q}, t)$ ] in terms of "Poisson bracket operator",

$$AB(p, q) = \exp \left[ \frac{\hbar}{2i} \left( \frac{\partial^{(a)}}{\partial p} \cdot \frac{\partial^{(b)}}{\partial q} - \frac{\partial^{(a)}}{\partial q} \cdot \frac{\partial^{(b)}}{\partial p} \right) \right] a(p, q) b(p, q), \quad (79)$$

or in terms of integral operator,

$$\begin{aligned} AB(p, q) &= \frac{1}{(2\pi\hbar)^8} \int dp' dq' K_A^+(p, q; p', q') b(p', q') \\ &= \frac{1}{(2\pi\hbar)^8} \int dp' dq' a(p', q') K_B^-(p, q; p', q') , \end{aligned} \quad (80)$$

where the factor  $\frac{1}{(2\pi\hbar)^8}$  accounts for the proper normalization of the integration (counting of states in terms of unit action) in  $(p, q, E, t)$ -space, and the integral kernels are defined by

$$K_Y^\pm(p, q; p', q') = \int du \, dv \exp \left\{ \frac{i}{\hbar} [(p - p') \cdot v + (q - q') \cdot u] \right\} y\left(p \pm \frac{u}{2}, q \mp \frac{v}{2}\right). \quad (81)$$

For numerical purposes using discrete lattice points, the following expressions of  $K_Y^\pm(p, q; p', q')$  is more preferable

$$K_Y^\pm(p, q; p', q') = \int du \, dv \exp \left\{ \frac{2i}{\hbar} [(p - p') \cdot v + (q - q') \cdot u] \right\} y(p \pm u, q \mp v). \quad (82)$$

Thus, we may write the LWtransform of a commutator  $[A, B]$  and an anticommutator  $\{A, B\}$  in terms of Poisson bracket differential operator,  $\Lambda$ , as

$$\begin{aligned} [A, B](p, q) &= \cos \Lambda [a(p, q) b(p, q) - b(p, q) a(p, q)] \\ &\quad - i \sin \Lambda \{a(p, q) b(p, q) + b(p, q) a(p, q)\}, \end{aligned} \quad (83)$$

$$\begin{aligned} \{A, B\}(p, q) &= \cos \Lambda \{a(p, q) b(p, q) + b(p, q) a(p, q)\} \\ &\quad - i \sin \Lambda [a(p, q) b(p, q) - b(p, q) a(p, q)], \end{aligned} \quad (84)$$

where  $\Lambda = \frac{\hbar}{2} \left( \frac{\partial^{(a)}}{\partial p} \cdot \frac{\partial^{(b)}}{\partial q} - \frac{\partial^{(a)}}{\partial q} \cdot \frac{\partial^{(b)}}{\partial p} \right)$ . In terms of integral operators, we have,

$$[A, B](p, q) = \frac{1}{(2\pi\hbar)^8} \int dp' dq' K_A^+(p, q; p', q') b(p', q') - b(p', q') K_A^-(p, q; p', q'), \quad (85)$$

$$\{A, B\}(p, q) = \frac{1}{(2\pi\hbar)^8} \int dp' dq' K_A^+(p, q; p', q') b(p', q') + b(p', q') K_A^-(p, q; p', q'). \quad (86)$$

The above expressions simplify considerably when the LWtransforms are scalar functions. For this case, we have the 'lattice' Weyl transform of a commutator and anti-commutator of two operators,  $[A, B]$  and  $\{A, B\}$ , respectively, given by the following expressions

$$[A, B](p, q) = \frac{1}{(2\pi\hbar)^8} \int dp' dq' K_A^s(p, q; p', q') b(p', q'), \quad (87)$$

$$\{A, B\}(p, q) = \frac{1}{(2\pi\hbar)^8} \int dp' dq' K_A^c(p, q; p', q') b(p', q'), \quad (88)$$

where,

$$\begin{aligned} K_Y^s(p, q; p', q') &= \int du dv \exp \left\{ \frac{2i}{\hbar} [(p - p') \cdot v + (q - q') \cdot u] \right\} \\ &\quad \times [y(p + u, q - v) - y(p - u, q + v)]. \end{aligned} \quad (89)$$

and

$$\begin{aligned} K_Y^c(p, q; p', q') &= \int du dv \exp \left\{ \frac{2i}{\hbar} [(p - p') \cdot v + (q - q') \cdot u] \right\} \\ &\quad \times [y(p + u, q - v) + y(p - u, q + v)]. \end{aligned} \quad (90)$$

## XI. CONCLUDING REMARKS

What we have accomplished in this paper is the demonstration of the formal mathematical structure of nonequilibrium multiband spin-correlation transport equations. It has been demonstrated that spin-dependent self-energies due to many-body effects give rise to torques in the system. For example, the many-body effects in spin-orbit coupling has lead to the separation of self-energy into spin-independent part and the corresponding spin vector.<sup>40,42,43</sup> Thus, it is expected that the results here will serve as a fundamental basis for constructing realistic transport equations, embodying various approximation schemes, for engineering expediency.

The range of their validity of all simpler and manageable approximate equations can be assessed in the light of the present mathematical structure given in this paper. This must be accurate enough for treating the space and time dependent spin relaxation and dephasing scattering mechanisms between conduction electrons, between valence holes, between electrons and holes, their coupling to pseudo-spin, and to the total charges. The total electron charges represented by  $S_{o,o}$  minus the positive background charge will be fed to the Poisson equation for self-consistency in the potential. It is expected that the accompanying physical interpretations will acquire deeper insights when applied to highly nonequilibrium situations for spintronic device-performance applications, based on the fundamental structure of SMQTEs presented in this paper, .

In particular, we have shown that the pseudo-spin is highly coupled to Pauli-Dirac spin transport equations and to the particle charge density. It is not clear how this pseudo-spin aspect of spin transport is described by conventional methods of Pauli-Dirac spin scattering physics so far employed in the literature on the theory of spin relaxation and dephasing mechanisms in solids.

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## Appendix A: Matrix Equations for Bloch Electrons with Spin

In the presence of spin degree of freedom, Eqs. (5), (12), (13), and (14) become matrix equations. For the conduction-electron band, Eq. (5), we have the matrix equation with spin indices given by,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \begin{pmatrix} G_{cc,\uparrow\uparrow}^{e-h,<}(\xi 2) & G_{cc,\uparrow\downarrow}^{e-h,<}(\xi 2) \\ G_{cc,\downarrow\uparrow}^{e-h,<}(\xi 2) & G_{cc,\downarrow\downarrow}^{e-h,<}(\xi 2) \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} v_{c,\uparrow\sigma}(1\xi) G_{cc,\sigma\uparrow}^{<}(\xi 2) \\ -G_{cc,\uparrow\sigma}^{<}(1\xi) v_{c,\sigma\uparrow}^T(\xi 2) \\ v_{c,\downarrow\sigma}(1\xi) G_{cc,\sigma\uparrow}^{<}(\xi 2) \\ -G_{cc,\downarrow\sigma}^{<}(1\xi) v_{c,\sigma\uparrow}^T(\xi 2) \end{bmatrix} & \begin{bmatrix} v_{c,\uparrow\sigma}(1\xi) G_{cc,\sigma\downarrow}^{<}(\xi 2) \\ -G_{cc,\uparrow\sigma}^{<}(1\xi) v_{c,\sigma\downarrow}^T(\xi 2) \\ v_{c,\downarrow\sigma}(1\xi) G_{cc,\sigma\downarrow}^{<}(\xi 2) \\ -G_{cc,\downarrow\sigma}^{<}(1\xi) v_{c,\sigma\downarrow}^T(\xi 2) \end{bmatrix} \\ + \begin{bmatrix} \Sigma_{cc,\uparrow\sigma}^r(1\xi) G_{cc,\sigma\uparrow}^{<}(\xi 2) \\ -G_{cc,\uparrow\sigma}^{<}(1\xi) \Sigma_{cc,\sigma\uparrow}^a(\xi 2) \\ \Sigma_{cc,\downarrow\sigma}^r(1\xi) G_{cc,\sigma\uparrow}^{<}(\xi 2) \\ -G_{cc,\downarrow\sigma}^{<}(1\xi) \Sigma_{cc,\sigma\uparrow}^a(\xi 2) \end{bmatrix} & \begin{bmatrix} \Sigma_{cc,\uparrow\sigma}^r(1\xi) G_{cc,\sigma\downarrow}^{<}(\xi 2) \\ -G_{cc,\uparrow\sigma}^{<}(1\xi) \Sigma_{cc,\sigma\downarrow}^a(\xi 2) \\ \Sigma_{cc,\downarrow\sigma}^r(1\xi) G_{cc,\sigma\downarrow}^{<}(\xi 2) \\ -G_{cc,\downarrow\sigma}^{<}(1\xi) \Sigma_{cc,\sigma\downarrow}^a(\xi 2) \end{bmatrix} \\ + \begin{bmatrix} \Sigma_{cc,\uparrow\sigma}^{<}(1\xi) G_{cc,\sigma\uparrow}^a(\xi 2) \\ -G_{cc,\uparrow\sigma}^r(1\xi) \Sigma_{cc,\sigma\uparrow}^{<}(\xi 2) \\ \Sigma_{cc,\downarrow\sigma}^{<}(1\xi) G_{cc,\sigma\uparrow}^a(\xi 2) \\ -G_{cc,\downarrow\sigma}^r(1\xi) \Sigma_{cc,\sigma\uparrow}^{<}(\xi 2) \end{bmatrix} & \begin{bmatrix} \Sigma_{cc,\uparrow\sigma}^{<}(1\xi) G_{cc,\sigma\downarrow}^a(\xi 2) \\ -G_{cc,\uparrow\sigma}^r(1\xi) \Sigma_{cc,\sigma\downarrow}^{<}(\xi 2) \\ \Sigma_{cc,\downarrow\sigma}^{<}(1\xi) G_{cc,\sigma\downarrow}^a(\xi 2) \\ -G_{cc,\downarrow\sigma}^r(1\xi) \Sigma_{cc,\sigma\downarrow}^{<}(\xi 2) \end{bmatrix} \\ + \begin{bmatrix} \Delta_{hh,cv,\uparrow\sigma}^{e-h,r}(1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,<}(\xi 2) \\ -g_{hh,cv,\uparrow\sigma}^{e-h,<}(1\xi) \Delta_{ee,vc,\sigma\uparrow}^{e-h,a}(\xi 2) \\ \Delta_{hh,cv,\downarrow\sigma}^{e-h,r}(1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,<}(\xi 2) \\ -g_{hh,cv,\downarrow\sigma}^{e-h,<}(1\xi) \Delta_{ee,vc,\sigma\uparrow}^{e-h,a}(\xi 2) \end{bmatrix} & \begin{bmatrix} \Delta_{hh,cv,\uparrow\sigma}^{e-h,r}(1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,<}(\xi 2) \\ -g_{hh,cv,\uparrow\sigma}^{e-h,<}(1\xi) \Delta_{ee,vc,\sigma\downarrow}^{e-h,a}(\xi 2) \\ \Delta_{hh,cv,\downarrow\sigma}^{e-h,r}(1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,<}(\xi 2) \\ -g_{hh,cv,\downarrow\sigma}^{e-h,<}(1\xi) \Delta_{ee,vc,\sigma\downarrow}^{e-h,a}(\xi 2) \end{bmatrix} \\ + \begin{bmatrix} \Delta_{hh,cv,\uparrow\sigma}^{e-h,<}(1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,a}(\xi 2) \\ -g_{hh,cv,\uparrow\sigma}^{e-h,r}(1\xi) \Delta_{ee,vc,\sigma\uparrow}^{e-h,<}(\xi 2) \\ \Delta_{hh,cv,\downarrow\sigma}^{e-h,<}(1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,a}(\xi 2) \\ -g_{hh,cv,\downarrow\sigma}^{e-h,r}(1\xi) \Delta_{ee,vc,\sigma\uparrow}^{e-h,<}(\xi 2) \end{bmatrix} & \begin{bmatrix} \Delta_{hh,cv,\uparrow\sigma}^{e-h,<}(1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,a}(\xi 2) \\ -g_{hh,cv,\uparrow\sigma}^{e-h,r}(1\xi) \Delta_{ee,vc,\sigma\downarrow}^{e-h,<}(\xi 2) \\ \Delta_{hh,cv,\downarrow\sigma}^{e-h,<}(1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,a}(\xi 2) \\ -g_{hh,cv,\downarrow\sigma}^{e-h,r}(1\xi) \Delta_{ee,vc,\sigma\downarrow}^{e-h,<}(\xi 2) \end{bmatrix} \end{pmatrix}, \quad (A1)
\end{aligned}$$

where the repeated spin subscript,  $\sigma$ , indicates the use of Einstein summation convention over the spin degrees of freedom.

For the hole band, we have the corresponding matrix equation determined from Eq. (12)

as,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \begin{pmatrix} G_{vv,\uparrow\uparrow}^{e-h,<}(\xi 2) & G_{vv,\uparrow\downarrow}^{e-h,<}(\xi 2) \\ G_{vv,\downarrow\uparrow}^{e-h,<}(\xi 2) & G_{vv,\downarrow\downarrow}^{e-h,<}(\xi 2) \end{pmatrix} \\
&= - \left( \begin{bmatrix} v_{vv,\uparrow\sigma}(1\xi) G_{vv,\sigma\uparrow}^{e-h,<}(\xi 2) \\ -G_{vv,\uparrow\sigma}^{e-h,<}(\xi 2) v_{vv,\sigma\uparrow}^T(\xi 2) \\ v_{vv,\downarrow\sigma}(1\xi) G_{vv,\sigma\uparrow}^{e-h,<}(\xi 2) \\ -G_{vv,\downarrow\sigma}^{e-h,<}(\xi 2) v_{vv,\sigma\uparrow}^T(\xi 2) \end{bmatrix} \begin{bmatrix} v_{vv,\uparrow\sigma}(1\xi) G_{vv,\sigma\downarrow}^{e-h,<}(\xi 2) \\ -G_{vv,\uparrow\sigma}^{e-h,<}(\xi 2) v_{vv,\sigma\downarrow}^T(\xi 2) \\ v_{vv,\downarrow\sigma}(1\xi) G_{vv,\sigma\downarrow}^{e-h,<}(\xi 2) \\ -G_{vv,\downarrow\sigma}^{e-h,<}(\xi 2) v_{vv,\sigma\downarrow}^T(\xi 2) \end{bmatrix} \right) \\
&- \left( \begin{bmatrix} \Sigma_{vv,\uparrow\sigma}^{e-h,r}(1\xi) G_{vv,\sigma\uparrow}^{e-h,<}(\xi 2) \\ -G_{vv,\uparrow\sigma}^{e-h,<}(\xi 2) \Sigma_{vv,\sigma\uparrow}^{e-h,a}(\xi 2) \\ \Sigma_{vv,\downarrow\sigma}^{e-h,r}(1\xi) G_{vv,\sigma\uparrow}^{e-h,<}(\xi 2) \\ -G_{vv,\downarrow\sigma}^{e-h,<}(\xi 2) \Sigma_{vv,\sigma\uparrow}^{e-h,a}(\xi 2) \end{bmatrix} \begin{bmatrix} \Sigma_{vv,\uparrow\sigma}^{e-h,r}(1\xi) G_{vv,\sigma\downarrow}^{e-h,<}(\xi 2) \\ -G_{vv,\uparrow\sigma}^{e-h,<}(\xi 2) \Sigma_{vv,\sigma\downarrow}^{e-h,a}(\xi 2) \\ \Sigma_{vv,\downarrow\sigma}^{e-h,r}(1\xi) G_{vv,\sigma\downarrow}^{e-h,<}(\xi 2) \\ -G_{vv,\downarrow\sigma}^{e-h,<}(\xi 2) \Sigma_{vv,\sigma\downarrow}^{e-h,a}(\xi 2) \end{bmatrix} \right) \\
&- \left( \begin{bmatrix} \Sigma_{vv,\uparrow\sigma}^{e-h,<}(1\xi) G_{vv,\sigma\uparrow}^{e-h,a}(\xi 2) \\ -G_{vv,\uparrow\sigma}^{e-h,r}(1\xi) \Sigma_{vv,\sigma\uparrow}^{e-h,<}(\xi 2) \\ \Sigma_{vv,\downarrow\sigma}^{e-h,<}(1\xi) G_{vv,\sigma\uparrow}^{e-h,a}(\xi 2) \\ -G_{vv,\downarrow\sigma}^{e-h,r}(1\xi) \Sigma_{vv,\sigma\uparrow}^{e-h,<}(\xi 2) \end{bmatrix} \begin{bmatrix} \Sigma_{vv,\uparrow\sigma}^{e-h,<}(1\xi) G_{vv,\sigma\downarrow}^{e-h,a}(\xi 2) \\ -G_{vv,\uparrow\sigma}^{e-h,r}(1\xi) \Sigma_{vv,\sigma\downarrow}^{e-h,<}(\xi 2) \\ \Sigma_{vv,\downarrow\sigma}^{e-h,<}(1\xi) G_{vv,\sigma\downarrow}^{e-h,a}(\xi 2) \\ -G_{vv,\downarrow\sigma}^{e-h,r}(1\xi) \Sigma_{vv,\sigma\downarrow}^{e-h,<}(\xi 2) \end{bmatrix} \right) \\
&- \left( \begin{bmatrix} \Delta_{ee,vc,\uparrow\sigma}^{e-h,r}(1\xi) g_{hh,cv,\sigma\uparrow}^{e-h,<}(\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,<}(\xi 2) \Delta_{hh,cv,\sigma\uparrow}^{e-h,a}(\xi 2) \\ \Delta_{ee,vc,\downarrow\sigma}^{e-h,r}(1\xi) g_{hh,cv,\sigma\uparrow}^{e-h,<}(\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,<}(\xi 2) \Delta_{hh,cv,\sigma\uparrow}^{e-h,a}(\xi 2) \end{bmatrix} \begin{bmatrix} \Delta_{ee,vc,\uparrow\sigma}^{e-h,r}(1\xi) g_{hh,cv,\sigma\downarrow}^{e-h,<}(\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,<}(\xi 2) \Delta_{hh,cv,\sigma\downarrow}^{e-h,a}(\xi 2) \\ \Delta_{ee,vc,\downarrow\sigma}^{e-h,r}(1\xi) g_{hh,cv,\sigma\downarrow}^{e-h,<}(\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,<}(\xi 2) \Delta_{hh,cv,\sigma\downarrow}^{e-h,a}(\xi 2) \end{bmatrix} \right) \\
&- \left( \begin{bmatrix} \Delta_{ee,vc,\uparrow\sigma}^{e-h,<}(1\xi) g_{hh,cv,\sigma\uparrow}^{e-h,a}(\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,r}(1\xi) \Delta_{hh,cv,\sigma\uparrow}^{e-h,<}(\xi 2) \\ \Delta_{ee,vc,\downarrow\sigma}^{e-h,<}(1\xi) g_{hh,cv,\sigma\uparrow}^{e-h,a}(\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,r}(1\xi) \Delta_{hh,cv,\sigma\uparrow}^{e-h,<}(\xi 2) \end{bmatrix} \begin{bmatrix} \Delta_{ee,vc,\uparrow\sigma}^{e-h,<}(1\xi) g_{hh,cv,\sigma\downarrow}^{e-h,a}(\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,r}(1\xi) \Delta_{hh,cv,\sigma\downarrow}^{e-h,<}(\xi 2) \\ \Delta_{ee,vc,\downarrow\sigma}^{e-h,<}(1\xi) g_{hh,cv,\sigma\downarrow}^{e-h,a}(\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,r}(1\xi) \Delta_{hh,cv,\sigma\downarrow}^{e-h,<}(\xi 2) \end{bmatrix} \right). \tag{A2}
\end{aligned}$$

## Appendix B: Spin-Dependent Pairing Green's Functions

For the 'pairing' between electron and holes, we have from Eq. (13)

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \begin{pmatrix} g_{hh,cv,\uparrow\uparrow}^{e-h,<} (12) & g_{hh,cv,\uparrow\downarrow}^{e-h,<} (12) \\ g_{hh,cv,\downarrow\uparrow}^{e-h,<} (12) & g_{hh,cv,\downarrow\downarrow}^{e-h,<} (12) \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} v_{cc,\uparrow\sigma} (\xi 2) g_{hh,cv,\sigma\uparrow}^{e-h,<} (1\xi) \\ -g_{hh,cv,\uparrow\sigma}^{e-h,<} (\xi 2) v_{vv,\sigma\uparrow}^T (1\xi) \\ v_{cc,\downarrow\sigma} (\xi 2) g_{hh,cv,\sigma\uparrow}^{e-h,<} (1\xi) \\ -g_{hh,cv,\downarrow\sigma}^{e-h,<} (\xi 2) v_{vv,\sigma\uparrow}^T (1\xi) \end{bmatrix} & \begin{bmatrix} v_{cc,\uparrow\sigma} (\xi 2) g_{hh,cv,\sigma\downarrow}^{e-h,<} (1\xi) \\ -g_{hh,cv,\uparrow\sigma}^{e-h,<} (\xi 2) v_{vv,\sigma\downarrow}^T (1\xi) \\ v_{cc,\downarrow\sigma} (\xi 2) g_{hh,cv,\sigma\downarrow}^{e-h,<} (1\xi) \\ -g_{hh,cv,\downarrow\sigma}^{e-h,<} (\xi 2) v_{vv,\sigma\downarrow}^T (1\xi) \end{bmatrix} \end{pmatrix} \\
&+ \begin{pmatrix} \begin{bmatrix} \Sigma_{cc,\uparrow\sigma}^{e-h,r} (1\xi) g_{hh,cv,\sigma\uparrow}^{e-h,<} (\xi 2) \\ -G_{cc,\uparrow\sigma}^{e-h,<} (1\xi) \Delta_{hh,cv,\sigma\uparrow}^{e-h,a} (\xi 2) \\ \Sigma_{cc,\downarrow\sigma}^{e-h,r} (1\xi) g_{hh,cv,\sigma\uparrow}^{e-h,<} (\xi 2) \\ -G_{cc,\downarrow\sigma}^{e-h,<} (1\xi) \Delta_{hh,cv,\sigma\uparrow}^{e-h,a} (\xi 2) \end{bmatrix} & \begin{bmatrix} \Sigma_{cc,\uparrow\sigma}^{e-h,r} (1\xi) g_{hh,cv,\sigma\downarrow}^{e-h,<} (\xi 2) \\ -G_{cc,\uparrow\sigma}^{e-h,<} (1\xi) \Delta_{hh,cv,\sigma\downarrow}^{e-h,a} (\xi 2) \\ \Sigma_{cc,\downarrow\sigma}^{e-h,r} (1\xi) g_{hh,cv,\sigma\downarrow}^{e-h,<} (\xi 2) \\ -G_{cc,\downarrow\sigma}^{e-h,<} (1\xi) \Delta_{hh,cv,\sigma\downarrow}^{e-h,a} (\xi 2) \end{bmatrix} \end{pmatrix} \\
&+ \begin{pmatrix} \begin{bmatrix} \Sigma_{cc,\uparrow\sigma}^{e-h,<} (1\xi) g_{hh,cv,\sigma\uparrow}^{e-h,a} (\xi 2) \\ -G_{cc,\uparrow\sigma}^{e-h,r} (1\xi) \Delta_{hh,cv,\sigma\uparrow}^{e-h,<} (\xi 2) \\ \Sigma_{cc,\downarrow\sigma}^{e-h,<} (1\xi) g_{hh,cv,\sigma\uparrow}^{e-h,a} (\xi 2) \\ -G_{cc,\downarrow\sigma}^{e-h,r} (1\xi) \Delta_{hh,cv,\sigma\uparrow}^{e-h,<} (\xi 2) \end{bmatrix} & \begin{bmatrix} \Sigma_{cc,\uparrow\sigma}^{e-h,<} (1\xi) g_{hh,cv,\sigma\downarrow}^{e-h,a} (\xi 2) \\ -G_{cc,\uparrow\sigma}^{e-h,r} (1\xi) \Delta_{hh,cv,\sigma\downarrow}^{e-h,<} (\xi 2) \\ \Sigma_{cc,\downarrow\sigma}^{e-h,<} (1\xi) g_{hh,cv,\sigma\downarrow}^{e-h,a} (\xi 2) \\ -G_{cc,\downarrow\sigma}^{e-h,r} (1\xi) \Delta_{hh,cv,\sigma\downarrow}^{e-h,<} (\xi 2) \end{bmatrix} \end{pmatrix} \\
&+ \begin{pmatrix} \begin{bmatrix} -\Delta_{hh,cv,\uparrow\sigma}^{e-h,r} (1\xi) G_{vv,\sigma\uparrow}^{e-h,>T} (\xi 2) \\ +g_{hh,cv,\uparrow\sigma}^{e-h,<} (1\xi) \Sigma_{vv,\sigma\uparrow}^{e-h,rT} (\xi 2) \\ -\Delta_{hh,cv,\downarrow\sigma}^{e-h,r} (1\xi) G_{vv,\sigma\uparrow}^{e-h,>T} (\xi 2) \\ +g_{hh,cv,\downarrow\sigma}^{e-h,<} (1\xi) \Sigma_{vv,\sigma\uparrow}^{e-h,rT} (\xi 2) \end{bmatrix} & \begin{bmatrix} -\Delta_{hh,cv,\uparrow\sigma}^{e-h,r} (1\xi) G_{vv,\sigma\downarrow}^{e-h,>T} (\xi 2) \\ +g_{hh,cv,\uparrow\sigma}^{e-h,<} (1\xi) \Sigma_{vv,\sigma\downarrow}^{e-h,rT} (\xi 2) \\ -\Delta_{hh,cv,\downarrow\sigma}^{e-h,r} (1\xi) G_{vv,\sigma\downarrow}^{e-h,>T} (\xi 2) \\ +g_{hh,cv,\downarrow\sigma}^{e-h,<} (1\xi) \Sigma_{vv,\sigma\downarrow}^{e-h,rT} (\xi 2) \end{bmatrix} \end{pmatrix} \\
&+ \begin{pmatrix} \begin{bmatrix} -\Delta_{hh,cv,\uparrow\sigma}^{e-h,<} (1\xi) G_{vv,\sigma\uparrow}^{e-h,rT} (\xi 2) \\ +g_{hh,cv,\uparrow\sigma}^{e-h,r} (1\xi) \Sigma_{vv,\sigma\uparrow}^{e-h,>T} (\xi 2) \\ -\Delta_{hh,cv,\downarrow\sigma}^{e-h,<} (1\xi) G_{vv,\sigma\uparrow}^{e-h,rT} (\xi 2) \\ +g_{hh,cv,\downarrow\sigma}^{e-h,r} (1\xi) \Sigma_{vv,\sigma\uparrow}^{e-h,>T} (\xi 2) \end{bmatrix} & \begin{bmatrix} -\Delta_{hh,cv,\uparrow\sigma}^{e-h,<} (1\xi) G_{vv,\sigma\downarrow}^{e-h,rT} (\xi 2) \\ +g_{hh,cv,\uparrow\sigma}^{e-h,r} (1\xi) \Sigma_{vv,\sigma\downarrow}^{e-h,>T} (\xi 2) \\ -\Delta_{hh,cv,\downarrow\sigma}^{e-h,<} (1\xi) G_{vv,\sigma\downarrow}^{e-h,rT} (\xi 2) \\ +g_{hh,cv,\downarrow\sigma}^{e-h,r} (1\xi) \Sigma_{vv,\sigma\downarrow}^{e-h,>T} (\xi 2) \end{bmatrix} \end{pmatrix}, \quad (B1)
\end{aligned}$$



and for its nonequilibrium (reverse) 'conjugate' process,<sup>44</sup> the matrix equation derived from Eq. (14), is,

$$\begin{aligned}
& i\hbar \left( \frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} \right) \begin{pmatrix} g_{ee,vc,\uparrow\uparrow}^{e-h,<} (1\xi) & g_{ee,vc,\uparrow\downarrow}^{e-h,<} (1\xi) \\ g_{ee,vc,\downarrow\uparrow}^{e-h,<} (1\xi) & g_{ee,vc,\downarrow\downarrow}^{e-h,<} (1\xi) \end{pmatrix} \\
&= \begin{pmatrix} \begin{bmatrix} v_{vv,\uparrow\sigma} (1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,<} (\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,<} (1\xi) v_{c,\sigma\uparrow}^T (\xi 2) \\ v_{vv,\downarrow\sigma} (1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,<} (\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,<} (1\xi) v_{c,\sigma\uparrow}^T (\xi 2) \end{bmatrix} & \begin{bmatrix} v_{vv,\uparrow\sigma} (1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,<} (\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,<} (1\xi) v_{c,\sigma\downarrow}^T (\xi 2) \\ v_{vv,\downarrow\sigma} (1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,<} (\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,<} (1\xi) v_{c,\sigma\downarrow}^T (\xi 2) \end{bmatrix} \\
+ \begin{bmatrix} -\Sigma_{vv,\uparrow\sigma}^{e-h,aT} (1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,<} (\xi 2) \\ +G_{vv,\uparrow\sigma}^{e-h,>T} (1\xi) \Delta_{ee,vc,\sigma\uparrow}^{e-h,a} (\xi 2) \\ -\Sigma_{vv,\downarrow\sigma}^{e-h,aT} (1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,<} (\xi 2) \\ +G_{vv,\downarrow\sigma}^{e-h,>T} (1\xi) \Delta_{ee,vc,\sigma\uparrow}^{e-h,a} (\xi 2) \end{bmatrix} & \begin{bmatrix} -\Sigma_{vv,\uparrow\sigma}^{e-h,aT} (1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,<} (\xi 2) \\ +G_{vv,\uparrow\sigma}^{e-h,>T} (1\xi) \Delta_{ee,vc,\sigma\downarrow}^{e-h,a} (\xi 2) \\ -\Sigma_{vv,\downarrow\sigma}^{e-h,aT} (1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,<} (\xi 2) \\ +G_{vv,\downarrow\sigma}^{e-h,>T} (1\xi) \Delta_{ee,vc,\sigma\downarrow}^{e-h,a} (\xi 2) \end{bmatrix} \\
+ \begin{bmatrix} -\Sigma_{vv,\uparrow\sigma}^{e-h,>T} (1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,a} (\xi 2) \\ +G_{vv,\uparrow\sigma}^{e-h,aT} (1\xi) \Delta_{ee,vc,\sigma\uparrow}^{e-h,<} (\xi 2) \\ -\Sigma_{vv,\downarrow\sigma}^{e-h,>T} (1\xi) g_{ee,vc,\sigma\uparrow}^{e-h,a} (\xi 2) \\ +G_{vv,\downarrow\sigma}^{e-h,aT} (1\xi) \Delta_{ee,vc,\sigma\uparrow}^{e-h,<} (\xi 2) \end{bmatrix} & \begin{bmatrix} -\Sigma_{vv,\uparrow\sigma}^{e-h,>T} (1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,a} (\xi 2) \\ +G_{vv,\uparrow\sigma}^{e-h,aT} (1\xi) \Delta_{ee,vc,\sigma\downarrow}^{e-h,<} (\xi 2) \\ -\Sigma_{vv,\downarrow\sigma}^{e-h,>T} (1\xi) g_{ee,vc,\sigma\downarrow}^{e-h,a} (\xi 2) \\ +G_{vv,\downarrow\sigma}^{e-h,aT} (1\xi) \Delta_{ee,vc,\sigma\downarrow}^{e-h,<} (\xi 2) \end{bmatrix} \\
+ \begin{bmatrix} \Delta_{ee,vc,\uparrow\sigma}^{e-h,r} (1\xi) G_{cc,\sigma\uparrow}^{e-h,<} (\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,<} (1\xi) \Sigma_{cc,\sigma\uparrow}^{e-h,a} (\xi 2) \\ \Delta_{ee,vc,\downarrow\sigma}^{e-h,r} (1\xi) G_{cc,\sigma\uparrow}^{e-h,<} (\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,<} (1\xi) \Sigma_{cc,\sigma\uparrow}^{e-h,a} (\xi 2) \end{bmatrix} & \begin{bmatrix} \Delta_{ee,vc,\uparrow\sigma}^{e-h,r} (1\xi) G_{cc,\sigma\downarrow}^{e-h,<} (\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,<} (1\xi) \Sigma_{cc,\sigma\downarrow}^{e-h,a} (\xi 2) \\ \Delta_{ee,vc,\downarrow\sigma}^{e-h,r} (1\xi) G_{cc,\sigma\downarrow}^{e-h,<} (\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,<} (1\xi) \Sigma_{cc,\sigma\downarrow}^{e-h,a} (\xi 2) \end{bmatrix} \\
+ \begin{bmatrix} \Delta_{ee,vc,\uparrow\sigma}^{e-h,<} (1\xi) G_{cc,\sigma\uparrow}^{e-h,a} (\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,r} (1\xi) \Sigma_{cc,\sigma\uparrow}^{e-h,<} (\xi 2) \\ \Delta_{ee,vc,\downarrow\sigma}^{e-h,<} (1\xi) G_{cc,\sigma\uparrow}^{e-h,a} (\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,r} (1\xi) \Sigma_{cc,\sigma\uparrow}^{e-h,<} (\xi 2) \end{bmatrix} & \begin{bmatrix} \Delta_{ee,vc,\uparrow\sigma}^{e-h,<} (1\xi) G_{cc,\sigma\downarrow}^{e-h,a} (\xi 2) \\ -g_{ee,vc,\uparrow\sigma}^{e-h,r} (1\xi) \Sigma_{cc,\sigma\downarrow}^{e-h,<} (\xi 2) \\ \Delta_{ee,vc,\downarrow\sigma}^{e-h,<} (1\xi) G_{cc,\sigma\downarrow}^{e-h,a} (\xi 2) \\ -g_{ee,vc,\downarrow\sigma}^{e-h,r} (1\xi) \Sigma_{cc,\sigma\downarrow}^{e-h,<} (\xi 2) \end{bmatrix} \end{pmatrix}. \tag{B2}
\end{aligned}$$

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- <sup>30</sup> This term is often attributed to electron *zitterbewegung* or fluctuating motion and is traditionally been neglected in the treatment of nonequilibrium quantum transport employed in practical applications.
- <sup>31</sup> This relation is used to bring Eqs. (43) and (44) to the electron picture. In fact this substitution is trivial by virtue of the locality in space and time so that the correlation functions involved are proportional to the absolute value of the field variables.
- <sup>32</sup> In the electron picture, the right-hand sides of Eqs. (42) and (45) have opposite sign, whereas in the electron-hole picture they have the same sign or have the same rate of change as discussed before.

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- <sup>35</sup> In the same sense perhaps where we usually treat the band structure as a separate calculations already done before we construct our two-band model of the semiconductor.
- <sup>36</sup> That the gradient expansion of SMQTEs, with the  $[\Sigma^<, \text{Re } S^r]$  term neglected, yields the classical Boltzmann transport equation for spinless system is often viewed as justifying the neglect of  $[\Sigma^<, \text{Re } S^r]$  in the quantum transport equations<sup>3</sup>.
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